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INFINITESIMAL VARIATION OF HYPERSURFACES OF A *GF* STRUCTURE MANIFOLD

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Abstract. In this paper we have studied the Infinitesimal Variations of the structure tensors of the generalized almost contact metric structure induced on the hypersurface of the *GF*-structure manifold under various conditions. Infinitesimal variations of the induced connexion and second fundamental form are used in order to deduce a few results.

1. **Introduction.** A differentiable manifold M^n equipped with a tensor field F of type (1,1) satisfying

$$F^2 = a^2 I, \tag{1.1}$$

where a is complex, real or imaginary and I is identity, is called a *GF*-structure manifold [1]. Further, on M^n a Riemannian metric G can be introduced satisfying

$$G(F\tilde{X}, F\tilde{Y}) = -a^2 G(\tilde{X}, \tilde{Y}), \tag{1.2}$$

where X, Y are arbitrary vector fields on M^n. Then the manifold is called an *H*-structure manifold [1].

Let us embed a hypersurface M^{n-1} into M^n by the isometric immersion $b: M^{n-1} \to M^n$. Correspondingly we have the Jacobian b^* of b, denoted by B. Since the immersion is isometric, we have [4]

$$G(BX, BY) = g(X, Y), \tag{1.3}$$

$$G(BX, N) = 0, \tag{1.4}$$

$$G(N, N) = 1, \tag{1.5}$$

Where g is the metric induced on the hypersurface M^{n-1}, X, Y denote arbitrary vector fields on M^{n-1} and N denotes the unit normal field on M^{n-1}.

We write the transformation equation as [3]

$$B\Phi X = FBX - A(X)N, \tag{1.6}$$

$$FN = -BT, \tag{1.7}$$

where Φ is a tensor field of type (1,1) and A is a 1-form on M^{n-1}. From (1.6) and (1.7) we obtain

$$(1.8)\quad \text{(a)}\quad \Phi^2 X = a^2 X + A(X)T, \qquad \text{(b)}\quad A(\Phi X) = 0,$$
$$\text{(c)}\quad A(T) = -a^2. \qquad \text{(d)}\quad \Phi T = 0.$$

The metric g is found to satisfy

$$\text{(e)}\quad g(\Phi X, \Phi Y) = -a^2 g(X, Y) - A(X)A(Y).$$

Let us call such a structure a *generalized almost contact metric structure*. Thus a generalized almost contact metric structure is induced on the hypersurface M^{n-1}.

If D is the Riemannian connexion induced on M^{n-1} by g, we have the Gauss and Weingarten formulae [3]

$$(1.9)\qquad E_{BX} BY = BD_X Y + h(X, Y) N,$$

$$(1.10)\qquad E_{BX} N = -BHX,$$

where h is the 2^{nd} fundamental form of M^{n-1} and H is a tensor field of type (1,1) associated to h. If K and $\tilde{K}$ stand for the curvature tensors of the hypersurface and the enveloping manifold, we have the Gauss and Codazzi equations [4]

$$(1.11)\qquad \tilde{K}(BX, BY, BZ, BU) = {}'K(X, Y, Z, U) - $$
$$- h(Y, Z)\, h(X, U) + h(X, Z)\, h(Y, U),$$

$$(1.12)\qquad {}'\tilde{K}(BX, BY, BZ, N) = (D_X h)(Y, Z) - (D_Y h)(X, Z),$$

where $'K$ and $'\tilde{K}$ are the associate covariant curvature tensors of M^{n-1} and M^n.

Now differentiating eqn. (1.6) along the hypersurface and using $E_{\tilde{X}} F = 0$ we get

$$E_{BY} B\Phi X = F(E_{BY} BX) - \{(D_Y A)X + A(D_Y X)\} N - A(X) E_{BY} N.$$

Using equations (1.7), (1.9) and (1.10) it reduces to

$$BD_Y \Phi X + h(Y, \Phi X) N = B\Phi D_Y X - h(Y, X) BT - N(D_Y A)X + A(X) BHY,$$

whose tangential and normal components are

$$(1.13)\qquad (D_Y \Phi) X = h(Y, X) T - A(X) HY,$$

$$(1.14)\qquad (D_Y A) X = -h(Y, \Phi X).$$

Covariant differentiation of the equation (1.7) along M^{n-1} gives

$$(1.15)\qquad D_X T = \Phi HX.$$

We now consider the tensor field $P(X, Y)$ defined by [7]

$$P(X, Y) = N(X, Y) + (dA)(X, Y)T,$$

where $N(X, Y)$ is the Nijenhious tensor formed with Φ. When the tensor field $P(X, Y)$ vanishes, the almost contact metric structure is said to be normal [7]. Thus the normality condition takes the form

$$P(X, Y) = (D_{\Phi X}\Phi)Y - (D_{\Phi Y}\Phi)X + \Phi(D_Y\Phi)X - \Phi(D_X\Phi)Y + \\ + (D_X A)Y - (D_Y A)XT = 0.$$

Using equations (1.13) and (1.14), we obtain

(1.16) $$H\Phi = \Phi H,$$

which implies that

(1.17) $$h(T, T) = HT.$$

Showing that $h(T, T)$ is an eigenvalue of H and the corresponding eigenvector is T. Let us denote $h(T, T)$ by τ.

The almost contact metric structure is called a contact structure if

(1.18) $$(D_X A)Y - (D_Y A)X = 2'\Phi(X, Y),$$

where $'\Phi(X, Y) = g(\Phi X, Y)$. But instead of (1.18) we assume [7]

(1.19) $$(D_X A)Y - (D_Y A)X = 2\alpha'\Phi(X, Y).$$

Applying (1.16) to the above equation, we have

(1.20) $$H\Phi = \Phi H = \alpha\Phi$$

whence we obtain

(1.21) $$HX = \alpha X + (\tau - \alpha)A(X)T.$$

When the almost contact metric structure is normal and contact, we call the structure a Sasakian structure [2]. Thus for a Sasakian structure equations (1.13), (1.14) and (1.15) have the form

(1.22) $$(D_X\Phi)Y = \alpha g(X, Y)T - \alpha A(Y)X,$$

(1.23) $$(D_X A)Y = \alpha'\Phi(X, Y),$$

(1.24) $$D_X T = \alpha\Phi X.$$

From (1.20) and (1.22) for a constant

(1.25) $$K(X, Y, T) = \alpha^2(A(X)Y - A(Y)X).$$

That is, the sectional curvature with respect to a plane section containing T is α^2.

2. Infinitesimal variation of a *GF*-structure manifold. Suppose that the infinitesimal variation of the hypersurface is brought about by the restriction of an almost decomposable killing vector field U on the enveloping manifold to the hypersurface. Accordingly the variation of the differential of embedding is given by [7]

$$(2.1)\qquad (\delta B)(X)=\varepsilon E_{BX}U,$$

where ε is infinitesimally small number. Split U into its tangential and normal components as $U=BV+\lambda N$. Using (1.9) and (1.10) we get

$$(2.2)\qquad (\delta B)(X)=\varepsilon\{B(D_X V-\lambda HX)+(X\lambda+h(X,V))N\}$$

Infinitesimal variation of N is given by [6]

$$\delta N=\varepsilon L_U N=\varepsilon BW,$$

$L_U N$ the Lie derivative of N being orthogonal to N. Infinitesimal variation of eqn. (1.4) yields

$$G(BD_X V+h(X,V)N+(X\lambda)N-\lambda BHX,N)=-G(BX,BW),$$

which implies $W=-(HV+\Lambda)$; Λ stands for the vector field associate to the gradient of λ. Thus we obtain

$$(2.3)\qquad \delta N=-\varepsilon B(HV+\Lambda).$$

Now varying eqn. (1.6) infinitesimally, we get

$$(\delta B)(\Phi X)+B(\delta\Phi)X=F((\delta B)X)-A(X)\,\delta N-N(\delta A)X.$$

Using (1.6) and (1.7) we find

$$B(\delta\Phi)X+(\delta A)(X)N=\varepsilon\{B(\Phi(D_X V-\lambda HX))+A(D_X V-\lambda HX)N-h(X,V)BT-$$
$$-X\lambda BT-BD_{\Phi X}V+\lambda BH\Phi X-h(\Phi X,V)N-(\Phi X)\lambda N+A(X)B(HV+\Lambda)\}.$$

Equating the tangential and normal parts, we get

$$(2.4)\qquad \begin{aligned}(\delta\Phi)X=\varepsilon\{&\Phi(D_X V-\lambda HX)-h(X,V)T-X\lambda T-D_{\Phi X}V+\\ &+\lambda H\Phi X+A(X)(HV+\Lambda)\},\end{aligned}$$

$$(2.5)\qquad (\delta A)X=\varepsilon\{A(D_X V-\lambda HX)-h(\Phi X,V)-(\Phi X)\lambda\}.$$

The Lie derivative of Φ along V is given by

$$(L_V\Phi)X=L_V(\Phi X)-\Phi(L_V X)=D_V(\Phi X)-D_{\Phi X}V-\Phi D_V X+\Phi D_X V.$$

Equation (2.4) assumes the form

$$(\delta\Phi)X=\varepsilon\{(L_V\Phi)X+\lambda(H\Phi-\Phi H)-h(X,V)T-X\lambda T+$$
$$+A(X)(HV+\Lambda)-(D_V\Phi)X\}.$$

Using (1.13) in the above equation we find

$$(\delta\Phi)X=\varepsilon\{(L_V\Phi)X+\lambda(H\Phi-\Phi H)X-h(X,V)T-X\lambda T+$$
$$+A(X)(HV+\Lambda)-h(X,V)T+A(X)HV=$$

(2.6) $$=\varepsilon\{(L_V\Phi)X+\lambda(H\Phi-\Phi H)X-2h(X,V)T+2A(X)HV+A(X)\Lambda\}$$

Applying eqn. (1.14) and the definition $(L_V A)X=(D_V A)X+A(D_X V)$ in (2.5) we get

(2.7) $$(\delta A)X=\varepsilon\{(L_V A)X-\lambda A(HX)-(\Phi X)\Lambda\}.$$

Next, varying eqn. (1.7) infinitesimally and using (1.6) and (2.2), we get

$$B\delta T+\varepsilon\{B(D_T V-\lambda HT)+(T\lambda+h(T,V))N-B\Phi(HV+\Lambda)-A(HV+\Lambda)N\}=0,$$

whose tangential part is

(2.8) $$\delta T=\varepsilon\{L_V T+\lambda HT+\Phi\Lambda\}.$$

Lastly, varying equation (1.3) infinitesimally we get

$$(\delta g)(X,Y)=G((\delta B)X,BY)+G(BX,(\delta B)Y)$$

which leads by virtue of eqn. (2.2) to

(2.9) $$(\delta g)(X,Y)=\varepsilon\{(L_V g)(X,Y)-2\lambda h(X,Y)\}$$

Thus we have the following theorem:

Theorem 2.1. *Under an infinitesimal variation* (2.1) *of a hypersurface of a GF-structure manifold, the variations of the structure tensors of the generalized almost contact metric structure induced on the hypersurface are given by*

(2.10)
$$(\delta\Phi)X=\varepsilon\{(L_V\Phi)X+\lambda(H\Phi-\Phi H)X-2h(X,V)T+$$
$$+2A(X)HV+A(X)\Lambda\},$$
$$(\delta A)X=\varepsilon\{(L_V A)X-\lambda A(HX)-(\Phi X)\Lambda\},$$
$$\delta T=\varepsilon\{L_V T+\lambda HT+\Phi\Lambda\},$$
$$(\delta g)(X,Y)=\varepsilon\{(L_V g)(X,Y)-2\lambda h(X,Y)\}.$$

Corollary 2.1. *When a hypersurface of a GF-structure manifold is given infinitesimal tangential variation by means of BV, the variations of the induced generalized almost contact metric structure tensors on the hypersurface are given by their Lie derivative along V.*

Corollary 2.2. *When a hypersurface of a GF-structure manifold is given infinitesimal normal variation by means of λN, the variations of the induced generalized almost contact metric structure tensors on the hypersurface are given by*

$$\text{(2.11)}\quad \begin{aligned} &\text{(a)} \quad (\delta\Phi)X = \varepsilon\{\lambda(H\Phi - \Phi H)X - 2h(X, V)T + 2A(X)HV + A(X)\Lambda\},\\ &\text{(b)} \quad (\delta A)X = \varepsilon\{\lambda A(HX) - (\Phi X)\Lambda\},\\ &\text{(c)} \quad \delta T = \varepsilon\{\lambda HT + \Phi\Lambda\},\\ &\text{(d)} \quad (\delta g)(X, Y) = \varepsilon\{-2\lambda h(X, Y)\}. \end{aligned}$$

The infinitesimal variation is parallel when $(\delta B)X$ is tangential to the original hypersurface. Since

$$(\delta B)X = \varepsilon\{B(D_X V - \lambda HX) + (X\lambda) + h(X, V)N\},$$

therefore for an infinitesimal parallel variation it is necessary and sufficient that $X\lambda + h(X, V) = 0$. Then an infinitesimal variation of a hypersurface will be parallel iff λ is constant.

Corollary 2.3. *When a hypersurface of a GF-structure manifold is given infinitesimal parallel variation, the structure tensors Φ, T, A and g of the hypersurface vary as*

$$\text{(2.12)}\quad \begin{aligned} &\text{(a)} \quad (\delta\Phi)X = \varepsilon\lambda(H\Phi - \Phi H)X, &&\text{(b)} \quad (\delta A)X = -\varepsilon\lambda A(HX),\\ &\text{(c)} \quad \delta T = \varepsilon\lambda HT, &&\text{(d)} \quad (\delta g)(X, Y) = -2\varepsilon\lambda h(X, Y). \end{aligned}$$

Corollary 2.4. *Let the structure induced on the hypersurface of a GF-structure manifold be a Sasakian structure with Φ sectional curvature $+\alpha^2$, then the infinitesimal normal parallel variation of the hypersurface makes the structure tensors vary as*

$$\text{(2.13)}\quad \begin{aligned} &\text{(a)} \quad (\delta\Phi)X = 0, \qquad \text{(b)} \quad (\delta A)X = -\varepsilon\lambda\tau T, \qquad \text{(c)} \quad \delta T = \varepsilon\lambda\tau T.\\ &\text{(d)} \quad (\delta g)(X, Y) = -2\varepsilon\lambda\{\alpha g(X, Y) + (\tau - \alpha)A(X)A(Y)\}. \end{aligned}$$

The infinitesimal variation of the connexion and the 2^{nd} fundamental form is given by [4]

$$\text{(2.14)}\quad \begin{aligned} (\delta D)(X, Y) = \varepsilon\{&(L_V D)(X, Y) - (D_Y\lambda H)X - (D_X\lambda H)Y +\\ &+ h(X, Y)\Lambda + \lambda h^*(X, Y)\}, \end{aligned}$$

$$\text{(2.15)}\quad \begin{aligned} (\delta h)(X, Y) = \varepsilon\{&(L_V h)(X, Y) - \lambda h(X, HY) + XY\lambda - (D_X Y) +\\ &+ \lambda' \tilde{K}(N, BX, BY, N)\}. \end{aligned}$$

Also if the infinitesimal variation of the hypersurface is normal, the variation of the induced connexion D and of the 2^{nd} fundamental form is given by [4]

$$(\delta D)(X, Y) = \varepsilon \{h(X, V)\Lambda + \lambda h^*(X, Y) - (D_Y \lambda H) X - (D_X \lambda H) Y, \tag{2.16}$$

$$(\delta h)(X, Y) = \varepsilon \{XY\lambda - (D_X Y)\lambda + '\tilde{K}(N, BX, BY, N) - \lambda h(X, HY)\}. \tag{2.17}$$

3. **Variation of a Sasakian hypersurface with Φ sectional curvature α^2.** Let the hypersurface bear a Sasakian structure with Φ sectional curvature α^2.

Theorem 3.1. *A Sasakian hypersurface with Φ sectional curvature α^2 will be varied infinitesimally to a Sasakian hypersurface with Φ sectional curvature $\alpha^2 + \delta\alpha^2$ iff λ satisfies the differential equation*

$$\begin{aligned} &\varepsilon [XY\lambda - (D_X Y)\lambda + \lambda \{'\tilde{K}(N, BX, BY, N) + \alpha^2 g(X, Y) - A(X) A(Y)\} + \\ &+ \{(D_T T)\lambda - (TT\lambda)\} A(X) A(Y) + (\tau - \alpha)\{(\Phi X)\lambda A(Y) + (\Phi Y)\lambda A(X)\}] = \\ &= \{A(X) A(Y) - g(X, Y)\}\,\delta\alpha. \end{aligned} \tag{3.1}$$

Proof. The Sasakian hypersurface with Φ sectional curvature α^2 will be varied infinitesimally to a Sasakian hypersurface with Φ sectional curvature $\alpha^2 + \delta\alpha^2$ iff

$$\begin{aligned} (\delta h)(X, Y) = (\delta\alpha)\, g(X, Y) &+ \alpha(\delta g)(X, Y) + \\ &+ \{(\delta h)(T, T) + 2h(T, \delta T) + \delta\alpha\} A(X) A(Y) + \\ &+ (\tau - \alpha)\{(\delta A)(X) A(Y) + (\delta A)(Y) A(X)\}, \end{aligned} \tag{3.2}$$

which with the help of equations (2.7), (2.8), (2.9), (2.19) and

$$\begin{aligned} (L_V h)(X, Y) = \alpha (L_V g)(X, Y) &+ \{(L_V h)(T, T) + 2h(L_V T, T)\} + \\ &+ (\tau - \alpha)\{(L_V A)(X) A(Y) + A(X)(L_V A)(Y)\} \end{aligned}$$

becomes

$$\begin{aligned} &-2\alpha\lambda h(X, Y) + \{2h(T, \lambda HT - \Phi\Lambda) + (1/\varepsilon)\,\delta\alpha + TT\lambda - \\ &-(D_T T)\lambda - \lambda h(T, HT)\} A(X) A(Y) - \\ &-(\tau - \alpha)\{\lambda A(HX) + (\Phi X)\Lambda) A(Y) + (\lambda A(HY) + (\Phi Y)\lambda) A(X)\} = \\ &= \varepsilon\{XY\lambda - (D_X Y)\lambda + \lambda' \tilde{K}(N, BX, BY, N) - \lambda h(X, HY)\}. \end{aligned}$$

We have $h(X, HY) = \alpha^2 g(X, Y) + (\tau^2 + \alpha^2) A(X) A(Y)$ and in particular $h(T, HT) = \tau^2$.

Consequently, the above condition reduces to (3.1).

Conversely, if λ satisfies the differential equation (3.1) then we get (3.2).

Corollary 3.1. *The infinitesimal parallel variation carries a Sasakian hypersurface with Φ-sectional curvature $+\alpha^2$ to a Sasakian hypersurface with Φ sectional curvature $+\alpha^2+\delta\alpha^2$ iff*

$$\lambda\varepsilon\{'\tilde{K}(N, BX, BY, N)+\alpha^2 g(X, Y)-A(X)A(Y)\}=$$

$$=\{A(X)A(Y)-g(X, Y)\}\,\delta\alpha. \tag{3.3}$$

Corollary 3.2. *If the enveloping manifold is flat, the condition reduces to*

$$\delta\alpha=-\lambda\varepsilon\alpha^2 \tag{3.4}$$

Corollary 3.3. *It is impossible to carry a Sasakian hypersurface (with Φ sectional curvature $+1$) of a flat GF-structure manifold over to a Sasakian hypersurface by an infinitesimal parallel variation.*

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