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A CONVERSE TO A GENERALIZED BANACH CONTRACTION PRINCIPLE

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Abstract. A converse to Jungck's generalization of Banach contraction principle has been derived.

Janos [1] and Meyers [2] have established converses to Banach contraction principle. The work of Janos was generalized by Edelstein [3] which was improved later by Park [4] in the light of the generalization of Banach contraction principle derived by Jungck [5]. In this note we derive a converse of Jungck's generalization of Banach contraction principle and, indeed, this converse generalizes Meyers'.

Meyers [2] has proved the following:

Theorem (*Meyers*). *Let X be a metrizable topological space and let its topology be generated by the metric ρ_0. Then for each $\lambda\in(0, 1)$ there exists a metric ρ_λ on X, complete if ρ_0 is complete, such that f is a ρ_λ-contraction iff (i) for some $\xi\in X$, $f\xi=\xi$, (ii) $f^n x\to\xi$ as $n\to\infty$ for all $x\in X$, (iii), there exists an open neighbourhood U of ξ such that $f^n(U)\to\{\xi\}$, which implies that given any neighbourhood V of ξ there is an integer $n(V)>0$ such that $f^n(U)\subset V$ for all $n\geqslant n(V)$.*

We now state our result as the following

Theorem. *Let X be a metrizable topological space whose topology is generated by a metric ρ_0. Let f be a homeomorphism of X onto itself and g a continuous self-map of X which commutes with f. Then for each $\lambda\in(0,1)$ there exists a metric ρ_λ, topologically equivalent to ρ_0, and complete if ρ_0 is complete, such that $\rho_\lambda(gx, gy)\leqslant\lambda\rho_\lambda(fx,fy)$, iff (a) there exists a point $\xi\in X$ such that $f\xi=g\xi=\xi$, (b) $f^{-n}g^n x\to\xi$, for all $x\in X$, (c) there exists a neighbourhood U of ξ, such that $f^{-n}g^n(U)\to\{\xi\}$, which means that given a neigbourhood V of ξ, there exits $n(V)>0$ such that for all $n\geqslant n(V) f^{-n}g^n(U)=(f^{-1}g)^n(U)\subset V$.*

Proof. *Necessity.* If such a metric ρ_λ exists, then since $g(X)\subset f(X)=X$ and $\rho_\lambda(gx, gy)\leqslant\lambda\rho_\lambda(fx,fy)$ we conclude, by Jungck's theorem, that f and g have a unique common fixed point ξ. Moreover, writing $fx=x'$, $fy=y'$, we have, $\rho_\lambda(f^{-1}gx', f^{-1}gy')\leqslant\lambda\rho_\lambda(x', y')$, whence $f^{-1}g$ is a Banach contraction, and (b) and (c) follow.

Sufficiency. Applying Meyers's theorem to the mapping $f^{-1}g$ we see that for each $\lambda \in (0, 1)$ there exists a metric ρ_λ, topologically equivalent to ρ_0, and complete if ρ_0 is, such that $\rho_\lambda(f^{-1}gx, f^{-1}gy) \leqslant \lambda \rho_\lambda(x, y)$ for all $x, y \in X$. Putting $f^{-1}x = x'$ and $f^{-1}y = y'$, we have, since f, g, f^{-1} commute, $\rho_\lambda(gx', gy') \leqslant \leqslant \lambda \rho_\lambda(fx', fy')$. Since f is a bijection, the above result implies that $\rho_\lambda(gx, gy) \leqslant \leqslant \lambda \rho_\lambda(fx, fy)$ for all $x, y \in X$. This completes the proof.

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