PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 32 (46) 1982, pp. 119—122

ABOUT THE FUNDAMENTAL MATRIX OF THE LINEAR NONSTATIONARY SYSTEM

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Abstract. The paper describes how the development of the fundamental matrix $\varphi(t, t_0)$ into the Taylor series and the method of solving differential equations by infinite series may be used for determining the fundamental matrix in the form of an infinite series. This method was compared with some other iterative methods and it was found that the results obtained with this method show greater accuracy.

If we observe a linear system of equations of the form

$$\dot{x}(t) = A(t) \cdot x(t) + B(t) \cdot u(t) \tag{1}$$

and if vector $x(t_0)$ is known, then by developing the fundamental matrix of system $\varphi(t, t_0)$ in the Taylor series and further applying the method of infinite series to the solution of differential equations, we may approximately determine the fundamental matrix with greater accuracy then by use of other iterative methods.

In equation system (1), $x(t)$ is an unknown phase column vector with coordinates x_i $(i = 1, 2, 3, \ldots, n)$, $A(t)$ is a known quadratic matrix of the n-th order, $u(t)$ a column vector with the u_k $(k = 1, 2, \ldots, r)$ known, and $B(t)$ is a known matrix of the (n, r) order.

The solution of equation system (1) is determined by the Cauchy formula

$$x(t) = \varphi(t, t_0) \cdot x(t_0) + \int_{t_0}^{t} \varphi(t, \tau) \cdot B(\tau) \cdot u(\tau)\, d\tau, \quad t \geqslant t_0 \tag{2}$$

where $\varphi(t, t_0)$ is the fundamental matrix of the system.

The application of formula (2) presupposes the knowledge of the fundamental matrix of the system. The defining of the fundamental matrix in its final form is not solved for all cases even in a stationary case. Hence, it is clear that in the non-stationary case this problem is very complex. Hitherto, only some approximate iterative methods were described in literature. One such method is reviewed here.

If the basic conditions are taken as a starting point for the fundamental matrix system $\varphi(t, t_0)$

$$\frac{\partial \varphi(t, t_0)}{\partial t} = A(t) \cdot \varphi(t, t_0) \tag{3}$$

(4) $$\varphi(t_0, t_0) = I \quad \text{(Unity matrix)}$$

the solution of equation (3) may be sought in the form

(5) $$\varphi(t, t_0) = \sum_{n=0}^{\infty} A_n(t_0) \frac{(t-t_0)^n}{n!}$$

where

(6) $$A_n(t_0) = \left. \frac{\partial^n \varphi(t, t_0)}{\partial t^n} \right|_{t=t_0}$$

the coefficients of the Taylor series (5) are not known and therefore should be determined. In order to determine the above-mentioned coefficients we shall substitute (5) in equation (3) and, at the same time, develop matrix $A(t)$ into the Taylor series. Equation (7) obtained as the result of these transformations:

(7) $$\sum_{n=1}^{\infty} A_n(t_0) \frac{(t-t_0)^{n-1}}{(n-1)!} = \left(\sum_{k=0}^{\infty} \overset{(k)}{A}(t_0) \frac{(t-t_0)^k}{k!} \right) \left(\sum_{k=0}^{\infty} A_k(t_0) \frac{(t-t_0)^k}{k!} \right)$$

has on the righthand side the product of two infinite series. On the basis of the Merteins theorem

(8) $$\left(\sum_{k=0}^{\infty} a_k \right) \left(\sum_{k=0}^{\infty} b_k \right) = \sum_{k=0}^{\infty} \sum_{\nu=0}^{k} a_\nu b_{k-\nu}$$

which may be applied under the condition that both series are convergent, one of them being absolutely convergent, equation (7) transforms into

(9) $$\sum_{k=0}^{\infty} A_{k+1}(t_0) \frac{(t-t_0)^k}{k!} = \sum_{k=0}^{\infty} \sum_{\nu=0}^{k} \overset{(\nu)}{A}(t_0) \frac{(t-t_0)^\nu}{\nu!} A_{k+\nu}(t_0) \frac{(t-t_0)^{k-\nu}}{(k-\nu)!} =$$

$$= \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \sum_{\nu=0}^{k} \binom{k}{\nu} \overset{(\nu)}{A}(t_0) A_{k-\nu}(t_0)$$

since

(10) $$A_{k+1}(t_0) = \sum_{\nu=0}^{k} \binom{k}{\nu} \overset{(\nu)}{A}(t_0) A_{k-\nu}(t_0), \quad k+1 = n$$

(11) $$A_n(t_0) = \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \overset{(\nu)}{A}(t_0) A_{n-\nu-1}(t_0) \quad n = 2, 3, 4, \dots$$

and here $\overset{(0)}{A} = A(t)$, $\overset{(1)}{A} = dA/dt$, $\overset{(\nu)}{A} = d^\nu A/dt^\nu$. From equation (6) we obtain that $A_0(t_0) = I$, $A_1(t_0) = A(t_0)$.

By substituting equation (11) into equation (5) we finally obtain

(12) $$\varphi(t, t_0) = I + (t-t_0) A(t_0) + \frac{(t-t_0)^2}{2!} (\dot{A} + A^2)_{t=t_0} +$$

$$+ \frac{(t-t_0)^3}{3!} (\ddot{A} + 2\dot{A}A + A\dot{A} + A^3)_{t=t_0} + \sum_{n=4}^{\infty} \left[\sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \overset{(\nu)}{A}(t_0) A_{n-\nu-1}(t_0) \right] \frac{(t-t_0)^n}{n!}$$

In the stationary case $A(t)=A=\text{Const.}$, i.e. its derivates are zero matrices, from equation (11) $A_n=A^n$ is obtained and hence

$$\varphi(t,\ t_0)=\sum_{n=0}^{\infty} A^n \frac{(t-t_0)^n}{n!}=e^{A(t-t_0)} \tag{13}$$

In the non-stationary case it is desirable to define the boundary function of series (12), by which fundamental matrix of the system is represented. If is not possible to determine the boundary function of the infinite series (12), an approximation should be made with first several terms. The error estimate has been defined in some special cases (error estimate may be made for all elements of the matrix as shown in the example given in the furher text).

Example. Let us assume that the observed object moves according to the equation $\ddot{y}+y\cdot\cos t=0$, where $y(t_0)$ is given. Determine the law of motion of the object $y(t)$ for $t>t_0$.

By the introduction of symbols $y=x_1$, $\dot{y}=x_2$ instead of a second order differential equation we obtain a system of equations of form (1), where now

$$A(t)=\begin{Vmatrix} 0 & 1 \\ -\cos t & 0 \end{Vmatrix},\ B(t)\cdot u(t)=0,\ t_0=0$$

We can determine $A_n(t_0)$ by equation (11) and substitute this into equation (12), thus obtaining

$$\varphi(t,\ 0)=\begin{Vmatrix} 1-t^2/2!+2\,t^4/4!-9\,t^6/6!+\cdots & t-t^3/3!+\cdots \\ -t+2\,t^3/3!-9\,t^5/5!+\cdots & 1-t^2/2!+\cdots \end{Vmatrix}$$

$$\begin{Vmatrix} y(t) \\ \dot{y}(t) \end{Vmatrix}=\begin{Vmatrix} x_1 \\ x_2 \end{Vmatrix}=\varphi(t,\ 0)\begin{Vmatrix} y_0 \\ \dot{y}_0 \end{Vmatrix}=x(t)$$

$$y(t)=(1-t^2/2!+2\,t^4/4!-9\,t^6/6!+\cdots)\,y_0+(t-t^3/3!+\cdots)\,\dot{y}_0$$

By the Leibnitz criterion, the elements of the matrix $\varphi(t,\ 0)$ are convergent, and the error is smaller than the neglected term.

If the interval is $t-t_0>1$, then the application of formula (2) is unsuitable because $(t-t_0)^n \xrightarrow[n\to\infty]{} \infty$ and hence the convergence of the series is slow. In this case the interval $(t-t_0)$ may be divided into m parts so that

$$t_0<t_1<\cdots t_k<t_{k+1}<\cdots<t_m=t,\ t_{k+1}-t_k=T,\ (k=0,\ 1,\ 2,\ \ldots,\ m)$$

where the value T is small.

On the basis of the property of the matrix

$$\varphi(t,\ t_0)=\varphi(t,\ t_{m-1})\,\varphi(t_{m-1},\ t_{m-2})\cdots\varphi(t_{k+1},\ t_k)\cdots\varphi(t_1,\ t_0) \tag{14}$$

where $\varphi(t_{k+1},\ t_k)=I+A(t_k)\cdot T+(\dot{A}+A^2)_{t=t_k}\,T^2/2!$

Conclusion. If the described procedure for determining the fundamental matrix is compared with that proposed in [3], some advantages of the described procedure are discovered. First of all, the derivation is a simpler mathematical operation than the integration and thus the applicability of expression (12) is much wider. In the case of discretisation, the error occurring may be estimated because it is proportional to $(t-t_0)^n$, which is not possible in the procedure illustrated in [3].

When solving simpler examples, it may be easily noted that the same result is obtained with both procedures if matrix elements $A(t)$ are polynomials. In more complex examples the comparison is more difficult because of the unsuitability of the procedure itself.

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