A NOTE ON GRAPHS REPRESENTABLE AS PRODUCTS OF GRAPHS

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1. Definitions. The ordered pair $G=(V(G), E(G))$, where $V(G)$ is a non-empty and finite set and where elements of $E(G)$ are subsets of $V(G)$ with two elements, is called a graph. The elements of $V(G)$ are vertices, while elements of $E(G)$ are edges of the graph $G$.

The complete graph and the cycle, with $n$ vertices, are denoted with $K_n$ and $C_n$. Specially, the vertices of the graph $K_2$, are denoted by 1 and 2.

The union $G_1 \cup G_2$ of graphs $G_1$ and $G_2$, is a graph $G$, where $V(G)=V(G_1) \cup V(G_2)$ and $E(G)=E(G_1) \cup E(G_2)$. If $V(G_i) \cap V(G_j)=\emptyset$, we use notation $G_1+G_2$ instead of $G_1 \cup G_2$. The graph $nG$ is the union $\sum_{i=1}^{n} G_i$, where $G_i \cong G$, $i=1, \ldots, n$.

The join $G_1 \vee G_2$ of graphs $G_1$ and $G_2$, $V(G_1) \cap V(G_2)=\emptyset$, is a graph $G$, where $V(G)=V(G_1) \cup V(G_2)$ and $\{x, y\} \subseteq E(G)$ if and only if $\{x, y\} \subseteq E(G_i)$ or $x \in V(G_i)$ and $y \in V(G_j)$, $i \neq j$, $i, j=1, 2$.

The product $G_1 \times G_2$ of graphs $G_1$ and $G_2$, is a graph $G$, where $V(G)=V(G_1) \times V(G_2)$ and $\{(x_1, y_1), (x_2, y_2)\} \subseteq E(G)$ if and only if $\{x_1, x_2\} \subseteq E(G_1)$ and $\{y_1, y_2\} \subseteq E(G_2)$.

2. Preliminaries. The product of graphs seems to have been first introduced by K. Čulik, who called it the cardinal product [4]. The product $G_1 \times G_2$ is also called conjunction, Kronecker product, tensor product etc.

P. Weichsel [8] has proved Theorem 1.

Theorem 1. The product $G_1 \times G_2$ of connected graphs $G_1$ and $G_2$ is connected if and only if either $G_1$ or $G_2$ contains an odd cycle.

Lemma 1 If $G$ is bipartite, then $K_2 \times G = 2G$.

![Fig. 1](image)

This lemma is noted in a paper by E. Sampathkumar [6]. D. Cvetković [1] mentions graphs $G_1$ and $G_2$, $\Gamma G_1 \cong G_2$, which satisfy the relation $K_2 \times G_1 \cong K_2 \times G_2$ (Fig. 1).
3. On the product $K_2 \times G$. It is possible to generalize the previous statement.

**Lemma 2.** For a given $k$, there are graphs $G, G_1, \ldots, G_k$, $G_i \neq G_j$ for $i \neq j$, for which $K_2 \times G_i = G$, $i = 1, \ldots, k$.

**Proof.** From $K_2 \times (2C_3) = K_2 \times C_6 = 2C_6$ we obtain $K_2 \times (2iC_3 + (k-i)C_6) = 2kC_6$ for $i = 1, \ldots, k$.

Let $V(G) = \{x_1, \ldots, x_n\}$, then $V(K_2 \times G) = \{(i, x) \mid i = 1, 2, j = 1, \ldots, n\}$. The graph $K_2 \times G$ is bipartite, the parts are $\{(1, x_i) \mid i = 1, \ldots, n\}$ and $\{(2, x_i) \mid i = 1, \ldots, n\}$. Both of them have the same number of vertices.

**Lemma 3.** The graph $K_2 \times G$ has at least one automorphism which matches the vertices with different coordinates.

**Proof.** Let $V(G) = \{x_1, \ldots, x_n\}$. The mapping $f : V(G) \to V(G)$ defined by $f(1, x_i) = (2, x_i), f(2, x_i) = (1, x_i)$ ($i = 1, \ldots, n$) is clearly an isomorphism.

**Lemma 4.** If $G$ a connected bipartite graph and if one component of the graph $K_2 \times G$ is isomorphic to $K_2 \times G$, then there is at least one isomorphism between any component of $K_2 \times G$ and $K_2 \times G_1$, which matches the vertices with equal first coordinates.

**Proof.** Let $G$ be a connected bipartite graph and let one component of $K_2 \times G$ be isomorphic to $K_2 \times G_1$. The graph $G_2 \times G_1$ has an automorphism which matches the vertices from different parts. Thus, any component of $K_2 \times G \cong 2G$ has an automorphism which matches the elements from different parts.

**Lemma 5.** If $K_2 \times G_1 \cong K_2 \times G_2$, then there is at least one isomorphism between $K_2 \times G_1$ and $K_2 \times G_2$, which matches the vertices with equal first coordinate.

**Proof.** Let $E_1, \ldots, E_m$ be components of the graph $G_1$, and let $F_1, \ldots, F_n$ be components of the graph $G_2$. Also, let $E_1, \ldots, E_r$, and $F_1, \ldots, F_s$ be bipartite graphs. Thus $E_1, E_2, E_3, \ldots, E_r, E_s, K_2 \times E_{r+1}, \ldots, K_2 \times E_m$ are the components of the graph $K_2 \times G_1$, while $F_1, F_2, F_3, \ldots, F_s, F_n, K_2 \times F_{s+1}, \ldots, K_2 \times F_m$ are the components of the graph $K_2 \times G_2$.

If $E_i \cong F_j$, for $i \leq r$ and $j \leq s$, or $K_2 \times E_i \cong K_2 \times F_j$, for $i > r$ and $j > s$, obviously we obtain Lemma 5. If $K_2 \times E_i \cong F_j$, for $i > r$ and $j \leq s$, or $E_i \cong K_2 \times F_j$, for $i \leq r$ and $j > s$, we obtain Lemma 5 from Lemma 4.

**Theorem 2.** If $K_2 \times G_1 \cong K_2 \times G_2$ and if $H$ is a bipartite graph, then $H \times G_1 \cong H \times G_2$.

**Proof.** Let $f$ be an isomorphism between $K_2 \times G_1$ and $K_2 \times G_2$, which matches vertices with equal first coordinate. Also, let $H$ have parts $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$. The mapping $g : V(H \times G_1) \to V(H \times G_2)$, defined by

\[
g(x_i, z) = (x_i, z_i), \text{ where } f(1, z) = (1, z_i),
\]

\[
g(y_j, z) = (y_j, z_2), \text{ where } f(2, z) = (2, z_2),
\]

for $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $z \in V(G_1)$, is clearly an isomorphism.
Theorem 3. If $K_2 \times G_1 = K_2 \times G_2$, then $K_2 \times (G_1 \oplus G) = K_2 \times (G_2 \oplus G)$ for each graph $G$.

Proof. Let $f$ be an isomorphism between $K_2 \times G_1$ and $K_2 \times G_2$, which matches vertices with equal first coordinate. The mapping $g$ defined by

$$g(x, y) = \begin{cases} f(x, y) & \text{if } y \in V(G_1) \\ (x, y) & \text{if } y \in V(G) \end{cases}$$

for $(x, y) \in V(K_2 \times (G_1 \oplus G))$, is clearly an isomorphism.

We give a characterization of the product $K_2 \times G$.

Definition 1. Let $G$ be a bipartite graph with bipartition $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\}$. Let $G^*$ be the graph with the following properties:

1. $(x_i, y_j) \in V(G^*)$ if and only if $\left\{\begin{array}{ll} x_i \in V(G_1) & \text{if } y_j \in V(G_2) \\ y_j \in V(G_2) & \text{if } x_i \in V(G_1) \end{array}\right.$

2. For all $i \neq i_2$ and $j \neq j_2$

$\{\{x_{i_1}, y_{j_1}\}, \{x_{i_2}, y_{j_2}\}\} \in E(G^*)$ if and only if $\{\{x_{i_1}, y_{j_1}\} \in E(G) \wedge \{x_{i_2}, y_{j_2}\} \in E(G)\}$.

Theorem 4. If $G$ is a bipartite graph with bipartition $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\}$, then there is a graph $G_1$ such that $G = K_2 \times G_1$ if and only if $G^*$ has a subgraph isomorphic to $K_n$.

Proof. Let $G = K_2 \times G_1$ and $V(G_1) = \{z_1, \ldots, z_n\}$. Thus, the subgraph of the graph $(K_2 \times G_1)^*$ with vertices $(1, z_i), (2, z_i) \mid i = 1, \ldots, n$ is isomorphic to $K_n$.

Let $G^*$ have a subgraph with vertices $(x_i, y_i), i = 1, \ldots, n$, which is isomorphic to $K_n$. Let $V(G_1) = \{(x_i, y_i) \mid i = 1, \ldots, n\}$ and $\{\{x_i, y_i\}, \{x_j, y_j\}\} \in E(G_1)$ if and only if $\{x_i, y_j\} \in E(G)$. Thus, $G = K_2 \times G_1$ with the isomorphism $f$ defined by

$f(x_i) = (1, (x_i, y_i)), f(y_j) = (2, (x_i, y_i)) \ (i = 1, \ldots, n)$.

4. A Graph Equation. Before we solve a graph equation, let's prove the following lemma.

Lemma 6. If a graph $G_1 \times G_2$ has the subgraph $K_n$, then $G_1$ and $G_2$ have the subgraph $K_n$.

Proof. Let the subgraph $K_n$ of a graph $G_1 \times G_2$ has the vertices $(x_i, y_i), \ i = 1, \ldots, n$. From $\{\{x_i, y_i\}, \{x_j, y_j\}\} \in E(G_1 \times G_2)$ we obtain $\{x_i, x_j\} \in E(G_1)$ and $\{y_i, y_j\} \in E(G_2)$. Thus, the graphs $G_1$ and $G_2$ have a subgraph isomorphic to $K_n$.

If $|V(G_1)| = 1$ and $G_1 \times G_2 = G_3 \times G_4$, then $|V(G_2)| = |V(G_3)| = V(G_4)| = 1$.

Theorem 5. If $|V(G_1)| \neq 1$ and

$$\overline{G_1 \times G_2} = G_3 \times G_4$$

then $G_i \cong K_1, i = 1, 2, 3, 4$. 
Proof. Let $V(G_1) = \{x_1, \ldots, x_n\}$ and $V(G_2) = \{y_1, \ldots, y_m\}$ and $n \geq m$. The vertices $(x_i, y_j)$, for $i = 1, \ldots, n$, are nonadjacent in $G_1 \times G_2$. Thus, $G_1 \times G_2$ has a subgraph $K_n$. From Lemma 5, we obtain $|V(G_1)| \cdot |V(G_2)| \geq n$. From $|V(G_1)| \cdot |V(G_2)| = |V(G_3)| \cdot |V(G_4)|$ we obtain $G_3 \cong G_4 \cong K_n$. Analogously, $G_1 \cong G_2 \cong K_n$.

The number of edges of a graph $K_n \times K_n$ is $2 \binom{n^2}{2}$. From (1) we get

$$\binom{n^2}{2} - 2 \binom{n}{2} = 2 \binom{n}{2}$$

or $n = 3$. The graphs $K_3 \times K_3$ and $K_3 \times K_3$ are given on Fig. 2., where an isomorphism is indicated.

![Fig. 2.](image)

**REFERENCES**


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