PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 32 (46), 1981, pp. 115—118

A NOTE ON GRAPHS REPRESENTABLE AS PRODUCTS OF GRAPHS

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1. Definitions. The ordered pair $G=(V(G), E(G))$, where $V(G)$ is a non-empty and finite set and where elements of $E(G)$ are subsets of $V(G)$ with two elements, is called a graph. The elements of $V(G)$ are vertices, while elements of $E(G)$ are edges of the graph G.

The complete graph and the cycle, with n vertices, are denoted with K_n and C_n. Specially, the vertices of the graph K_2, are denoted by 1 and 2.

The union $G_1 \cup G_2$ of graphs G_1, and G_2, is a graph G, where $V(G)=V(G_1)\cup V(G_2)$ and $E(G)=E(G_1)\cup E(G_2)$. If $V(G_1)\cap V(G_2)=\varnothing$, we use notation G_1+G_2 instead of $G_1\cup G_2$. The graph nG is the union $\sum_{i=1}^{n} G_i$, where $G_i \cong G$, $i=1,\ldots,n$.

The join $G_1 \nabla G_2$ of graphs G_1 and G_2, $V(G_1)\cap V(G_2)=\varnothing$, is a graph G, where $V(G)=V(G_1)\cup V(G_2)$ and $\{x, y\}\in E(G)$ if and only if $\{x, y\}\in E(G_i)$ or $x\in V(G_i)$ and $y\in V(G_j)$ $i\neq j$ $i, j=1, 2$.

The product $G_1\times G_2$ of graphs G_1 and G_2 is a graph G, where $V(G)=V(G_1)\times V(G_2)$ and $\{(x_1\, y_1), (x_2\, y_2)\}\in E(G)$ if and only if $\{x_1\, x_2\}\in E(G_1)$ and $\{y_1\, y_2\}\in E(G_2)$.

2. Preliminaries. The product of graphs seems to have been first introduced by K. Čulik, who called it the cardinal product [4]. The product $G_1\times G_2$ is also called conjunction, Kronecker product, tensor product etc.

P. Weichsel [8] has proved Theorem 1.

Theorem 1. *The product $G_1\times G_2$ of connected graphs G_1 and G_2 is connected if and only if either G_1 or G_2 contains an odd cycle.*

Lemma 1 *If G is bipartite, then $K_2\times G=2G$.*

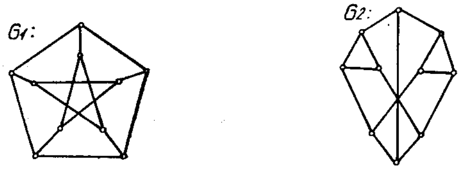

Fig. 1

This lemma is noted in a paper by E. Sampathkumar [6]. D. Cvetković [1] mentions graphs G_1 and G_2, $\neg\, G_1\cong G_2$, which satisfy the relation $K_2\times G_1\cong K_2\times G_2$ (Fig. 1).

3. On the product $K_2 \times G$. It is possible to generalize the previous statement.

Lemma 2. *For a given k, there are graphs $G, G_1, \ldots, G_k$, $G_i \neq G_j$ for $i \neq j$,* for *which $K_2 \times G_i = G$, $i = 1, \ldots, k$.*

Proof. From $K_2 \times (2C_3) = K_2 \times C_6 = 2C_6$ we obtain $K_2 \times (2iC_3 + (k-i)\,C_6) = 2kC_6$ for $i = 1, \ldots, k$.

Let $V(G) = \{x_1, \ldots, x_n\}$, then $V(K_2 \times G) = \{(i, x_j)\ i = 1, 2.\ j = 1, \ldots, n\}$. The graph $K_2 \times G$ is bipartite, the parts are $\{(1, x_i) \mid i = 1, \ldots, n\}$ and $\{2, x_i) \mid i = 1, \ldots, n\}$. Both of them have the same number of vertices.

Lemma 3. *The graph $K_2 \times G$ has at least one automorphism which matches the vertices with different first coordinates.*

Proof. Let $V(G) = \{x_1, \ldots, x_n\}$. The mapping $f: V(G) \to V(G)$ defined by $f(1, x_i) = (2, x_i)$, $f(2, x_i) = (1, x_i)$ $(i = 1, \ldots, n)$ is clearly an isomorphism.

Lemma 4. *If G a connected bipartite graph and if one component of the graph $K_2 \times G$ is isomorphic to $K_2 \times G$, then there is at least one isomorphism between any component of $K_2 \times G$ and $K_2 \times G_1$, which matches the vertices with equal first coordinates.*

Proof. Let G be a connected bipartite graph and let one component of $K_2 \times G$ be isomorphic to $K_2 \times G_1$. The graph $G_2 \times G_1$ has an automorphism which matches the vertices from different parts. Thus, any component of $K_2 \times G \cong 2G$ has an automoiphism which matches the elements from diffcrent parts.

Lemma 5 *If $K_2 \times G_1 \cong K_2 \times G_2$, then there is at least one isomorphism between $K_2 \times G_1$ and $K_2 \times G_2$, which matches the vertces with equal first coordinate.*

Proof. Let $E_1, \ldots, E_m$ be components of the graph G_1 and let $F_1, \ldots, F_n$ be components of the graph G_2. Also, let $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$ be bipartite graphs. Thus $E_1, E_1, E_2. E_2, \ldots, E_r, E_r$, $K_2 \times E_{r+1}, \ldots, K_2 \times E_m$ are the components of the graph $K_2 \times G_1$, while $F_1, F_1, F_2, F_2, \ldots, F_s, F_s, K_2 \times F_{s+1}, \ldots, K_2 \times F_n$ are the components of the graph $K_2 \times G_2$.

If $E_i \cong F_j$, for $i \leqslant r$ and $j \leqslant s$, or $K_2 \times E_i \cong K_2 \times F_j$, for $i > r$ and $j > s$, obviously we obtain Lemma 5. If $K_2 \times E_i \cong F_i$, for $i > r$ and $j \leqslant s$, or $E_i \cong K_2 \times F_j$, for $i \leqslant r$ and $j > s$, we obtain Lemma 5 from Lemma 4.

Theorem 2. *If $K_2 \times G_1 \cong K_2 \times G_2$ and if H is a bipartite graph, then $H \times G_1 \cong H \times G_2$.*

Proof. Let f be an isomorphism between $K_2 \times G_1$ and $K_2 \times G$, which matches vertices with equal first coordinate. Also, let H have parts $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$. The mapping g: $V(H \times G_1) \to V(H \times G_2)$, defined by

$$g(x_i, z) = (x_i, z_1), \text{ where } f(1, z) = (1, z_1),$$

$$g(y_j, z) = (y_j, z_2), \text{ where } f(2, z) = (2, z_2),$$

for $i = 1, \ldots, m$, $j = 1, \ldots, n$ and $z \in V(G_1)$, is clearly an isomorphism.

Theorem 3. *If $K_2 \times G_1 = K_2 \times G_2$, then $K_2 \times (G_1 \nabla G) = K_2 \times (G_2 \nabla G)$ for each graph G.*

Proof. Let f be an isomorphism between $K_2 \times G_1$ and $K_2 \times G_2$, which matches vertices with equal first coordinate. The mapping g defined by

$$g(x, y) = \begin{cases} f(x, y) & \text{if } y \in V(G_1) \\ (x, y) & \text{if } y \in V(G) \end{cases}$$

for $(x, y) \in V(K_2 \times (G_1 \nabla G))$, is clearly an isomorphism.

We give a characterization of the product $K_2 \times G$.

Definition 1. Let G be a bipartite graph with bipartition $\{x_1, \ldots, x_n\} \cup \cup \{y_1, \ldots, y_n\}$. Let G^* be the graph with the following properties:

1. $(x_i, y_j) \in V(G^*) \Leftrightarrow \{x_i\, y_j\} \in E(G)$, 2. for all $i_1 \neq i_2$ and $j_1 \neq j_2$

$$\{(x_{i_1}, y_{j_1}), (x_{i_2}, y_{j_2})\} \in E(G^*) \Leftrightarrow (\{x_{i_1}, y_{j_2}\} \in E(G) \wedge \{x_{i_2}, y_{j_1}\} \in E(G)).$$

Theorem 4. *If G is a bipartite graph with bipartition $\{x_1, \ldots, x_n\} \cup \cup \{y_1, \ldots, y_n\}$, then there is a graph G_1 such that $G = K_2 \times G_1$ if and only if G^* has a subgraph isomorphic to K_n*

Proof. Let $G = K_2 \times G_1$ and $V(G_1) = \{z_1, \ldots, z_n\}$. Thus, the subgraph of the graph $(K_2 \times G_1)^*$ with vertices $\{(1, z_i), (2, z_i) \mid i = 1, \ldots, n\}$ is isomorphic to K_n.

Let G^* has a subgraph with vertices $(x_i\, y_i),$, $i = 1, \ldots, n$, which is isomorphic to K_n. Let $V(G_1) = \{(x_i, y_i) \mid i = 1, \ldots, n\}$ and $\{(x_i, y_i), (x_j, y_j)\} \in E(G_1)$ if and only if $\{x_i\, y_j\} \in E(G)$. Thus, $G = K_2 \times G_1$ with the isomorphism f defined by

$$f(x_i) = (1, (x_i, y_i)),\ f(y_i) = (2, (x_i, y_i))\ \ (i = 1, \ldots, n).$$

4. A Graph Equation. Before we solve a graph equation, let's prove the following lemma.

Lemma 6. *If a graph $G_1 \times G_2$ has the subgraph K_n, then G_1 and G_2 have the subgraph K_n.*

Proof. Let the subgraph K_n of a graph $G_1 \times G_2$ has the vertices (x_i, y_i), $i = 1, \ldots, n$. From $\{(x_i, y_i), (x_j, y_j)\} \in E(G_1 \times G_2)$ we obtain $\{x_i, x_j\} \in \in E(G_1)$ and $\{y_i, y_j\} \in E(G_2)$. Thus, the graphs G_1 and G_2 have a subgraph isomorphic to K_n.

If $|V(G_1)| = 1$ and $\overline{G_1 \times G_2} = G_3 \times G_4$, then $|V(G_2)| = |V(G_3)| = V(G_4)| = 1$.

Theorem 5. *If $|V(G_1)| \neq 1$ and*

$$\overline{G_1 \times G_2} = G_3 \times G_4 \tag{1}$$

then $G_i \cong K_3$, $i = 1, 2, 3, 4$.

P r o o f. Let $V(G_1)=\{x_1, \dots, x_n\}$ and $V(G_2)=\{y_1, \dots, y_m\}$ and $n \geqslant m$. The vetrices (x_i, y_1), for $i=1, \dots, n$, are nonadjacent in $G_1 \times G_2$. Thus, $\overline{G_1 \times G_2}$ has a subgraph K_n. From Lemma 5. we obtain $|V(G_3)|$, $|V(G_4)| \geqslant n$. From $|V(G_1)| \cdot \cdot |V(G_2)| = |V(G_3)| \cdot V|(G_4)|$ we obtain $G_3 \cong G_4 \cong K_n$. Analogously, $G_1 \cong G_2 \cong K_n$. The number of edges of a graph $K_n \times K_n$ is $2\binom{n}{2}^2$. From (1) we get

$$\binom{n^2}{2} - 2\binom{n}{2}^2 = 2\binom{n}{2}^2$$

or $n=3$. The graphs $K_3 \times K_3$ and $\overline{K_3 \times K_3}$ are given on Fig. 2., where an isomorphism is indicated.

$K_3 \times K_3$ Fig. 2. $\overline{K_3 \times K_3}$

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(Received 13 04 1081)