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SOME REMARKS ON POINT PICARD SETS FOR ENTIRE FUNCTIONS

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1. Introduction and statement of results. Following Lehto [4] we call a plane set E a Picard set for entire functions if every entire transcendental function $f(z)$ has the following property: For every finite value a, with at most one exception, $f(z)$ takes a infinitely often in the complement CE of E. This is the only sense in which we shall use the expression "Picard set" in what follows.

From this definition, if E is not a Picard set, there must be an entire transcendental function $f(z)$, such that for two distinct values a and b, $f(z)$ takes a and b at most finitely often in CE. Clearly we can assume without loss of generality that a and b are 0 and 1 respectively.

Recent results on Picard sets in this sense may be found in Toppila [6] and [7] and also in Baker and Liverpool [1] and [2]. In Lehto [4] the following result was proved.

A set $E=\{a_n\}$ $n=1, 2, 3, \dots$ *is a Picard set for entire function if* $|a_n/a_{n+1}|=O(1/n^2)$.

By restricting the points of E to lie on a ray the above condition can be weakened to $|a_{n+1}/a_n|\geqslant q>1$.

In this paper, we shall prove in a direct manner that the restriction "points of E lie on a ray" is not necessary. Ideed the condition $|a_{n+1}/a_n|\geqslant q>1$ is sufficient to ensure our point set $E=\{a_n\}$ is a Picard set. We observe also that the entire function $f(z)=\left(1+\operatorname{Cos}\sqrt{z}\right)/2$ takes 0 and 1 at the points $a_n=n^2\pi^2$, $n=0, 1, 2, \dots$. This shows that the condition $|a_{n+1}/a_n|>1+2/n$ is not strong enough to give us Picard sets. However for this function, and its zero and one points, $|a_{n+1}/a_n|\to 1$ as $n\to\infty$. We show that we can indeed obtain Picard sets $E=\{a_n\}$ for which $|a_{n+1}/a_n|\to 1$ as $n\to\infty$. More precisely, we shall prove the following theorem.

Theorem 1. *If* $E=\{a_n\}$, $n=1, 2, 3, \dots$, *is a countable point set and* $|a_{n+1}/a_n|\geqslant q>1$ *then* $E=\{a_n\}$ *is a Picard set for entire transcendental functions. Moreover, we can obtain Picard sets* $E=\{a_n\}$ *for which* $|a_n|<|a_{n+1}|\to\infty$ *and* $|a_{n+1}/a_n|\to 1$ *as* $n\to\infty$. *In particular, any set with* $|a_n|=\exp(\delta n/\log n)$, $\delta>k\pi^2$, $k>1$ *is such a set.*

After establishing this theorem, we shall briefly discuss our result in the light of other related results in the literature.

2. Conditions for a sequence of points $\{a_n\}$ to be a Picard set. We quote the following statement of Schottky's theorem.

Lemma 1. (Hayman [3]) *If $f(z)$ is regular and satisfies $f(z)\neq 0$, 1 on $|z|<1$, then $M(r, f)\leqslant\Omega(a_0, r)$, where $r=|z|<1$ and $\Omega(a_0, r)$ satisfies*

$$\Omega(a_0, r)\leqslant\exp(1, r)^{-1}\{(1+r)\log\max(1, |a_0|)+2Cr\}$$

Here C is an absolute constant and $a_0=f(0)$. We now state and prove our second lemma — a result which we shall need in the proof of our theorem

Lemma 2. *Suppose $f(z)$ is an entire function and $\{a_n\}$ is a sequence of complex numbers with $|a_n|<|a_{n+1}|$, and $|a_n|\to\infty$ with n. If all the solutions of the equations, $f(z)=0$ and $f(z)=1$ occur at the points a_n, then there exists arbitrarily large n and points z_n, with $|z_n|=\sqrt{|a_n a_{n+1}|}$ such that $|f(z_n)|<2$.*

Proof. Let $\alpha=0$ or 1 be taken at most a finite number of times by $f(z)$. Then there is a path Γ leading to infinity on which $f(z)\to\alpha$.

The points at which Γ cuts the circle $|z|=\sqrt{|a_n a_{n+1}|}$ for n large enough satisfy the assertion of the lemma. We now assume that $f(z)$ takes 0 and 1 infinitely often. We consider an arbitrarily large z_0 such that $f(z_0)=1$. This z_0 must be one of our a_n. There is now a curve γ through z_0 on which $|f(z)|=1$. There are two possibilities! Either γ is not closed but extends to infinity, in which case our argument above applies or, γ is closed, enclosing a region D inside which $|f(z)|<1$ in which case D contains a zero a_n of $f(z)$.

Now, we let $z_0=a_m$, $a_m\neq a_n$. There is a path σ (in D appart from one end point at z_0) joining a_n and z_0 and this path meets either $|z|=\sqrt{|a_m a_{m-1}|}$ or $|z|=\sqrt{|a_m a_{m+1}|}$ at a point where $|f(z)|<1$.

Since $|z_0|=|a_m|$ may be chosen arbitrarily large, the lemma is proved. We now go on to apply this lemma to prove our theorem.

Theorem 1. *If $|a_{n+1}/a_n|\geqslant q>1$ then $E=\{a_n\}$, $n=1, 2, 3, \ldots$ is a Picard set for entire transcendental functions. Moreover we can obtain Picard sets $E=\{a_n\}$ in which $|a_n|<|a_{n+1}|\to\infty$ and $|a_{n+1}/a_n|\to 1$ as $n\to\infty$. In particular any countable point set with $|a_n|=\exp(\delta n/\log n)$ with $\delta>k\pi^2$, $k>1$ is such a set.*

Proof. Under our hypothesis $E=\{a_n\}$, with $|a_n|<|a_{n+1}|\to\infty$. We suppose now that E is not a Picard set for entire transcendental functions. Then there must be an entire transcendental function $f(z)$, whose zeros and ones lie entirely in E. We apply our lemma and hence there are arbitrarily large n, such that the annulus $A: |a_n|<|z|<|a_{n+1}|$ contains no zeros or ones of $f(z)$ and there is a z_n, with $|z_n|=\sqrt{|a_n a_{n+1}|}$ at which $|f(z_n)|<2$. We suppose this

z_n lies on the real axis. We map the annulus A by the transformation $z=e^t$ to a strip.

$$\log\alpha=\log|a_n|<\text{Re}\,t<\log|a_{n+1}|=\log\rho$$

We then introduce the transformation $u=c+dt$ with $c=-d\log z_n$ and $d=$ $=\pi/(2\log\rho/\alpha)$.

This takes our first strip into a second one $-\pi/4<\text{Re}\,u<\pi/4$.

Finally we introduce the transformation $w=\tan u$ to map this second strip onto the unit disc $|w|<1$.

Thus the chain of transformations mentioned above, i.e.

$$z=e^t,\ u=c+dt,\ c=d\log z_n,\ d=\pi/(2\log\rho/\alpha)\ \ w=\tan u$$

maps the annulus A onto the unit disc with z_n mapping to the origin $w=0$ and the circle $|z|=|z_n|=\sqrt{|a_n a_{n+1}|}$ described just once maps into the segment of the imaginary axis between $\pm i\tau$, where

$$\tau=\tanh[\pi^2/(2\log\beta/\alpha)]=1-2\{\exp[\pi^2/(\log\rho/\alpha)]+1\}^{-1}$$

We now apply Lemma 1, to our annulus A to get

$$M(r,\ g)<\exp(1-r)^{-1}\{(1+r)\log 2+2Cr\}$$

For $r=|w|<1$, this becomes

$$M(r,\ g)<\exp k^1(1-r)^{-1},\ k^1\ \text{an absolute constant.}$$

Here $f(z)$ in our annulus is now regarded as a function $g(w)$ in $|w|<1$ with $g(w)\neq 0$, 1 there. Also $|a_0|=|g(0)|=|f(z_n)|<2$. In particular for values of $f(z)$ on $|z|=\rho=\sqrt{|a_n a_{n+1}|}$ we have

$$M(\rho,\ f)=\max_{|y|\leqslant\tau}|g(iy)|<\exp(2\log 2+2c)/(1-\tau))$$

Thus $|f(z)|<\exp\{k^1/2\cdot(\exp[\pi^2/(\log\beta/\alpha)]+1)\}$. (A)

Using our coefficient condition in the hypothesis, $|a_{n+1}/a_n|\geqslant q>1$ i.e. $\rho/\alpha>1$ we see that $|f(z)|<k_1$, a bound independent of n, for z satisfying $|z|=\sqrt{|a_n a_{n+1}|}=\rho$. But this contradicts Liouville's theorem and hence E must be a Picard set for entiere transcendental functions, thus establishing our theorem.

Let us now take $|a_n|=\exp(\delta n/\log n)$, $\delta>k\pi^2$, $k>1$. Then $\log|a_{n+1}/a_n|\sim$ $\sim\delta/\log n$ which converges to zero when $n\to\infty$. Thus $\log|a_{n+1}/a_n|>\delta/k\log n$ and

$$\pi^2/\log|a_{n+1}/a_m|<(k\pi^2\log n)/\delta<\sigma\log n,\ \text{with}\ \ \sigma=k\pi^2/\delta<1.$$

Using (A) and this inequality, we get for large n that

$$|f(z)|<\exp\{k^1(e^{\sigma\log n}+1)/2\}<A\exp(k^1 n^\sigma/2)<\exp(\delta n/\log n),\ \ A=\exp k^1/2$$

Thus on $|z|=\rho=\sqrt{|a_n a_{n+1}|}$, $|f(z)|<|a_n|<\rho$. This again contradicts Liouville's theorem since $f(z)$ is entire transcendental. Hence our proof is complete.

3. Related Results in the Literature. Let us look at two results in the literature which throw some more light on our theorem.

Matsumoto [5] showed that if $f(z)$ is an entire transcendental function, then $f(z)$ takes every finite value outside a set E infinitely often except for at most one when the points a_n, of E satisfy the condition $\log|a_{n+1}/a_n| \geqq m(n)$ where $m(n)$, $n=1, 2, 3, \ldots..$ are positive numbers such that

$$\overline{\lim_{t\to\infty}} \left(K^{1/m(t)} \Big/ \sum_{n=1}^{t} m(n) \right) < \infty, \ K \text{ a const.}$$

In the considerations in the proof of our theorem, we used an explicit choice of a_n, but in general, our method works provided on $|z| = \rho = \sqrt{|a_n a_{n+1}|}$

$$\log|f(z)| < k'/2 \cdot (\exp(\pi^2/\log|a_{n+1}/a_n) + 1)$$

implies $\log|f(z)| < k_1 \log \rho$ and hence $f(z)$ has to be a polynomial contradicting Liouville's theorem. This is indeed the case if

$\exp(\pi^2/\log|a_{n+1}|a_n|) + 1 < k'' \log|a_n|$ that is $\exp(\pi^2/\log|a_{n+1}/a_n|) < k_2 \log|a_n|$.

Now, let us put $M(n) = \log|a_{n+1}/a_n|$. We notice that

$$\log|a_n| = \log\left|\frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} a_1\right| = \sum_{t-1}^{n-1} m(t) + \log|a_1|$$

Thus we see that our method works as long as $\exp(\pi^2/m(n)) \Big/ \sum_{t=1}^{n-1} m(t)$ is bounded for all arbitrarily large n.

Topilla [7] has also shown that a countable point set $E = \{a_n\}$ whose points converge to infinity is a Picard set of entire functions if there exists $\varepsilon > 0$, such that

$$\{z \mid 0 < |z - a_n| < \log \varepsilon |a_n| / \log|a_n|\} \cap E = \varnothing$$

for all sufficiently large n.

In the same paper Matsumoto shows shat his result is best possible in the sense that corresponding to each realvalued function $h(r)$ satisfying $h(r) \to \infty$ as $r \to \infty$, there exists a countable set $E = \{a_n\}$ whose points converge to infinity which is not a Picard set for entire functions, and which satisfies the condition

$$\{z \mid 0 < |z - a_n| < |a_n| / h(|a_n| \log|a_n|)\} \cap E = \varnothing$$

for all sufficiently large n. In our theorem our points are so far apart, that Matsumoto's second condition is satisfied.

Let us finally comment that our theorem gives a direct proof that the restriction which appears in Lehto's paper is not neccesary. It also displays explicitly that we can have Picard sets $E = \{a_n\}$ such that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ despite the function $f(z) = (1 + \cos\sqrt{z})/2$ which shows that $E = \{a_n\}$ with $a_n = n^2 \pi^2$ is not a Picard set.

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