KNESER'S THEOREM FOR WEAK SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper we consider the differential equation \( x' = f(t, x), \ x(0) = x_0, \) where \( f \) is a weakly-weakly continuous function from \([0, a] \times B\) into a Banach space \( E \) and \( B \) is a ball in \( E \). It is shown that if \( \beta(f([0, a] \times \mathcal{X})) < h(\beta(X)) \) for each \( X \subseteq B \), where \( \beta \) is a measure of weak noncompactness and \( h \) is a Kacze function, then the set of all weak solutions of this equation, defined on a compact interval \( J \), is continuum in the space \( C_w(J, E) \) of weakly continuous functions \( J \to E \) provided with the topology of weak uniform convergence.

Assume that \( J = [0, a], E \) is a Banach space with the norm \( \| \cdot \|, B = \{ x \in E : \| x - x_0 \| \leq b \} \), \( f : I \times B \to E \) is weakly-weakly continuous and \( \| f(t, x) \| \leq M \) on \( I \times B \); moreover, assume that \( E_w \) – the space \( E \) endowed with the weak topology, is sequentially weakly complete. Let \( J = [0, d] \), where \( d = \min(a, b/M) \).

We shall deal with the Cauchy problem

\[
(CP) \quad x' = f(t, x), \ x(0) = x_0.
\]

A function \( x : J \to B \) is called a solution of \( (CP) \) if

a) \( x \) is weakly differentiable on \( J \);

b) \( x(0) = x_0 \);

c) \( x'(t) = f(t, x(t)) \) for \( t \in J \), where \( x' \) denotes the weak derivative of \( x \).

It is known [8] that \( (CP) \) is equivalent to the integral equation

\[
x(t) = x_0 + \int_0^t f(s, x(s)) \, ds \quad \text{for} \quad t \in J,
\]

where \( \int \) denotes the Pettis integral.

Denote by \( C_w(J, E) \) the space of weakly continuous functions \( u : J \to E \) endowed with the topology of weak uniform convergence.

In this paper using measure of weak noncompactness developed by de Blasi [2] we prove that the set of all solutions of the Cauchy problem \( (CP) \) is nonempty, compact and connected in \( C_w(J, E) \).
Definition. Let $A$ be a bounded nonvoid subset of $E$. The measure of weak noncompactness $\beta (A)$ is defined by $\beta (A) = \inf \{ t > 0 : \text{there exists } C \subseteq K^w \text{ such that } A \subseteq C + tB_0 \}$, where $K^w$ is the set of weakly compact subsets of $E$ and $B_0$ is the norm unit ball.

The properties of weak noncompactness $\beta$ are analogous to the properties of measure of noncompactness:

1) if $A \subseteq B$ then $\beta (A) \leq \beta (B)$;

2) $\beta (A^w) = \beta (\tilde{A}^w)$ where $\tilde{A}^w$ denotes the weak closure of $A$;

3) $\beta (A) = 0$ if and only if $\tilde{A}^w$ is weakly compact;

4) $\beta (A \cup B) = \max \{ \beta (A), \beta (B) \}$;

5) $\beta (A) = \beta (C \circ A)$;

6) $\beta (A + B) \leq \beta (A) + \beta (B)$;

7) $\beta (x + A) = \beta (A)$ where $x \in E$;

8) $\beta (tA) = t \beta (A)$, $t > 0$;

9) $\beta (A) \leq \beta (A)$ (the diameter of $A$).

We shall use the following lemmas:

Lemma 1. For any $h \in E^*$, $\varepsilon > 0$ and for any weakly continuous function $y : J \to B$ there exists a weak neighbourhood $U$ of $0$ in $E$ such that

$$\int_0^d | h(f(s, x(s)) - f(s, y(s))) | ds \leq \varepsilon$$

for every weakly continuous function $x : J \to B$ such that $x(t) - y(t) \in U$ for each $t \in J$.

Lemma 2. Let $H \subseteq C_w(J, E)$ be a family of strongly equicontinuous functions. Then $\beta (H(J)) = \sup_{t \in J} \beta (H(t))$ where $H(J) = \{ u(t) : u \in H, t \in J \}$.

Proof. Since $H(t) \subseteq H(J)$, $\beta (H(t)) \leq \beta (H(J))$, we have $\sup_{t \in J} \beta (H(t)) \leq \beta (H(J))$. In order to prove the converse inequality suppose that for $\varepsilon > 0$, $0 = t_0 < t_1 < \cdots < t_n = d$ is a partition of $[0, d]$ such that $\| u(t) - u(s) \| \leq \varepsilon$ for every $t, s \in [t_i, t_{i+1}]$, $i = 0, 1, \ldots, n - 1$, and $u \in H$.

It follows from the definition of $\beta$ that for any $i$ there exists a weakly compact subset $P_i$ such that $H(t_i) \subseteq P_i + (\varepsilon + \beta (H(t_i)))B_0$. Now for $t \in J$ and $u \in H$ we have

$$u(t) = u(t_i) + (u(t) - u(t_i)) \subseteq P_i + (\varepsilon + \beta (H(t_i)))B_0 + \varepsilon B_0 \subseteq$$

$$\subseteq P + (2\varepsilon + \beta (H(t_i)))B_0 \subseteq P + (2\varepsilon + \sup_{t \in J} \beta (H(t)))B_0$$

where $P = \bigcup_{i=1}^n P_i$. This implies that $\beta (H(J)) \leq 2 \varepsilon + \sup_{t \in J} \beta (H(t))$. As the above inequality holds for any $\varepsilon > 0$, we get $\beta (H(J)) \leq \sup_{t \in J} \beta (H(t))$, which ends the proof.
Kneser's theorem for weak solutions of ordinary differential equations in Banach spaces

Given any $\eta > 0$ let us denote by $S_\eta$ the set of all functions $u: J \to E$ such that $u(0) = x_0$, $\| u(t) - u(s) \| \leq M | t - s |$ for $t, s \in J$ and

$$\sup_{t \in J} \| u(t) - x_0 - \int_0^t f(s, u(s)) \, ds \| < \eta.$$ 

In [9] it is shown that $S_\eta$ is a nonempty and connected subset of $C_w(J, E)$.

Now we prove a Kneser type theorem for the problem (CP). Assume that a nonnegative real-valued function $h$ is non-decreasing on $\mathbb{R}_+$ and $u(t) = 0$ is the unique solution of the integral equation

$$z(t) = \int_0^t h(z(s)) \, ds \text{ on } J.$$ 

**Theorem.** *If for every subset $X$ of $B$

(1) $\beta(f(J \times X)) \leq h(\beta(X))$,

then the set $S$ of all weak solutions of the Cauchy problem (CP) defined on $J$ is nonempty, compact and connected in $C_w(J, E)$.*

**Proof.** Let $\tilde{B}$ denote the set of all weakly continuous functions $x: J \to B$. Put

$$F(x)(t) = x_0 + \int_0^t f(s, x(s)) \, ds \text{ for } t \in J \text{ and } x \in \tilde{B}.$$ 

It is clear from Lemma 1 that $F$ is a continuous mapping of $\tilde{B}$ into $C_w(J, E)$.

1. First we shall show that the set $S$ is nonempty. For any $n \in N$ we choose $u_n \in S_{1/n}$. Let $H = \{ u_n : n \in N \}$.

It follows from the definition of $S_\eta$ that $\| u_n(t) - u_n(s) \| \leq M | t - s |$ for $n = 1, 2, \ldots$ and $t, s \in J$. Hence $| \beta(H(t)) - \beta(H(s)) | \leq 2M | t - s |$, which proves the continuity of the function $t \to \nu(t) = \beta(H(t))$ on $J$.

For fixed $t \in J$ we divide the interval $[0, t]$ into $m$ parts: $0 = t_0 < t_1 < \cdots < t_m = t$, where $t_i = \alpha | t | / m$, $i = 0, 1, \ldots, m$. Let

$$H([t_{i-1}, t_i]) = \{ u(s) : u \in H, \ t_{i-1} \leq s \leq t_i \}.$$ 

By Lemma 2 and the continuity of $\nu$ there is $s_i \in [t_{i-1}, t_i]$ such that

(2) $\beta(H([t_{i-1}, t_i])) = \sup \{ \beta(H(s)) : t_{i-1} \leq s \leq t_i \} = \nu(s_i)$.

On the other hand, by the mean value theorem [1; Th. V. 10. 4.] we obtain

$$F(u)(t) = x_0 + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} f(s, u(s)) \, ds \in x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i) \text{ conv } f(J \times H([t_i, t_{i+1}]))$$

for each $u \in H$. Therefore

$$F(H)(t) \subseteq x_0 + \sum_{i=0}^{m-1} (t_{i+1} - t_i) \text{ conv } f(J \times H([t_i, t_{i+1}])).$$
By (1), (2) and the corresponding properties of \( \beta \) it follows that
\[
\beta(F(H)(t)) \leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) \beta(f(J \times H([t_i, t_{i+1}])))
\]
\[
\leq \sum_{i=0}^{m-1} (t_{i+1} - t_i) h(\beta(H([t_i, t_{i+1}]))) = \sum_{i=0}^{u-1} (t_{i+1} - t_i) h(v(s_i)).
\]
But if \( m \to \infty \), then
\[
\sum_{i=0}^{m-1} (t_{i+1} - t_i) h(v(s_i)) \to \int_0^t h(v(s)) \, ds.
\]
Thus
\[
(3) \quad \beta(F(H)(t)) \leq \int_0^t h(v(s)) \, ds \text{ for } t \in J.
\]
Because \( \lim_{n \to \infty} \|u_n - F(u_n)\|_c = 0 \) and
\[
H(t) \subseteq \{u_n(t) - F(u_n)(t) : u_n \in H\} + F(H)(t),
\]
so \( \beta(H(t)) \leq \beta(F(H)(t)) \), and finally, by (3)
\[
v(t) \leq \int_0^t h(v(s)) \, ds \text{ for } t \in J.
\]
Applying now theorem on differential inequalities (cf. [6]) we get \( v(t) = \beta(H(t)) = 0 \) for each \( t \in J \). Consequently \( H(t) \) is conditionally weakly compact. Hence Ascoli's Theorem [7] proves that \( H \) is conditionally compact in \( C_w(J, E) \). Therefore the sequence \( (u_n) \) has a limit point \( u \).

Since \( \lim (u_n - F(u_n)) = 0 \) and \( F \) is continuous, we obtain that \( u = F(u) \), i.e. \( u \in S \).

2. Now we shall prove that the set \( S \) is compact.

Since \( F \) is continuous, \( S \) is closed in \( C_w(J, E) \). As \( S = F(S) \), by repeating the argument from 1 we can show that \( S \) is conditionally compact in \( C_w(J, E) \).

Suppose that the set \( S \) is not connected. As \( S \) is compact, thus there are nonempty compact sets \( W_1 \) and \( W_2 \) such that \( S = W_1 \cup W_2 \) and \( W_1 \cap W_2 = \emptyset \), and consequently there are two disjoint open sets \( U_1, U_2 \) such that \( W_1 \subseteq U_1 \), \( W_2 \subseteq U_2 \). Suppose that for every \( n \in \mathbb{N} \) there exists a \( u_n \in V_n \setminus U \), where \( V_n = S_{1/n} \) and \( U = U_1 \cup U_2 \). Put \( H = \{u_n : n \in \mathbb{N}\} \). Since \( u_n - F(u_n) \to 0 \) as \( n \to \infty \), by repeating the argument from 1 we see that there exists \( u_0 \in H \) such that \( u_0 = F(u_0) \), i.e. \( u_0 \in S \). Furthermore \( S \subseteq C_w(J, E) \setminus U \), since \( U \) is open, so that \( u_0 \in S \setminus U \), a contradiction.

Therefore there is \( m \in \mathbb{N} \) such that \( V_m \subseteq U \). Since \( U_1 \cap V_m \neq \emptyset \neq U_2 \cap V_m \), this shows that \( V_m \) is not connected, in contradiction with the connectedness of all \( V_n \). Hence \( S \) is connected.

Remark. The assumption (I) was used by Evin Cramer, V. Lakshmikantham and A.R. Mitchel in [4] in the proof of existence theorem, and is analogous to the corresponding assumptions for measure of noncompactness (cf. [3], [5]).
Kneser’s theorem for weak solutions of ordinary differential equations...

REFERENCES


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