PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 32 (46), 1982, pp. 99–103

KNESER'S THEOREM FOR WEAK SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. In this paper we consider the differential equation $x'=f(t,x)$, $x(0)=x_0$, where f is a weakly-weakly continuous function from $[0, a]\times B$ into a Banach space E and B is a ball in E. It is shown that if $\beta(f([0, a]\times X))\leqslant h(\beta(X))$ for each $X\subset B$, where β is a measure of weak noncompactness and h is a Kamke function, then the set of all weak solutions of this equation, defined on a compact interval J, is continuum in the space $C_w(J, E)$ of weakly continuous functions $J\to E$ provided with the topology of weak uniform convergence.

Assume that $I=[0, a]$, E is a Banach space with the norm $\|\cdot\|$, $B=\{x\in E:\|x-x_0\|\leqslant b\}$, $f:I\times B\to E$ is weakly-weakly continuons and $\|f(t, x)\|\leqslant M$ on $I\times B$; moreover, assume that E_w – the space E endowed with the weak topology, is sequentially weakly complete. Let $J=[0, d]$, where $d=\min(a, b/M)$.

We shall deal with the Cauchy problem

(CP) $$x'=f(t, x),\ x(0)=x_0.$$

A function $x:J\to B$ is called a solution of (CP) if

a) x is weakly differentiable on J;

b) $x(0)=x_0$;

c) $x'(t)=f(t, x(t))$ for $t\in J$, where x' denotes the weak derivative of x.

It is known [8] that (CP) is equivalent to the integral equation

$$x(t)=x_0+\int_0^t f(s, x(s))\, ds \quad \text{for } t\in J,$$

where $\int$ denotes the Pettis integral.

Denote by $C_w(J, E)$ the space of weakly continuous functions $u:J\to E$ endowed with the topology of weak uniform convergence.

In this paper using measure of weak noncompactness developed by de Blasi [2] we prove that the set of all solutions of the Cauchy problem (CP) is nonempty, compact and connected in $C_w(J, E)$.

Definition. Let A be a bounded nonvoid subset of E. The measure of weak noncompactness $\beta(A)$ is defined by $\beta(A)=\inf\{t>0:$ there exists $C\in K^w$ such that $A\subset C+tB_0\}$, where K^w is the set of weakly compact subsets of E and B_0 is the norm unit ball.

The properties of weak noncompactness β are analogous to the properties of mesure of noncompactness:

1) if $A\subset B$ then $\beta(A)\leqslant\beta(B)$;

2) $\beta(A)=\beta(\bar{A}^w)$ where $\bar{A}^w$ denotes the weak closure of A;

3) $\beta(A)=0$ if and only if $\bar{A}^w$ is weakly compact;

4) $\beta(A\cup B)=\max\{\beta(A), \beta(B)\}$;

5) $\beta(A)=\beta(C\circ A)$;

6) $\beta(A+B)\leqslant\beta(A)+\beta(B)$;

7) $\beta(x+A)=\beta(A)$ where $x\in E$;

8) $\beta(tA)=t\beta(A)$, $t>0$;

9) $\beta(A)\leqslant\delta(A)$ (the diameter of A).

We shall use the following lemmas:

Lemma 1. *For any $h\in E^*$, $\varepsilon>0$ and for any weakly continuous function $y:J\to B$ there exists a weak neighbourhood U of 0 in E such that*

$$\int_0^d |h(f(s, x(s))-f(s, y(s)))|\,ds\leqslant\varepsilon$$

for every weakly continuous function $x:J\to B$ such that $x(t)-y(t)\in U$ for each $t\in J$.

Lemma 2. *Let $H\subset C_w(J, E)$ be a family of strongly equicontinuous functions. Then $\beta(H(J))=\sup_{t\in J}\beta(H(t))$ where $H(J)=\{u(t):u\in H,\ t\in J\}$.*

Proof. Since $H(t)\subset H(J)$, $\beta(H(t))\leqslant\beta(H(J))$, we hawe $\sup_{t\in J}\beta(H(t))\leqslant$ $\leqslant\beta(H(J))$. In order to prove the converse inequality suppose that for $\varepsilon>0$, $0=t_0<t_1<\cdots<t_n=d$ is a partition of $[0, d]$ such that $\|u(t)-u(s)\|\leqslant\varepsilon$ for every $t, s\in[t_i, t_{i+1}]$, $i=0,1,\ldots,n-1$, and $u\in H$.

It follows from the definition of β that for any i there exists a weakly compact subset P_i such that $H(t_i)\subset P_i+(\varepsilon+\beta(H(t_i)))B_0$. Now for $t\in J$ and $u\in H$ we have

$$u(t)=u(t_i)+(u(t)-u(t_i))\in P_i+(\varepsilon+\beta(H(t_i)))B_0+\varepsilon B_0\subset$$
$$\subset P+(2\varepsilon+\beta(H(t_i)))B_0\subset P+(2\varepsilon+\sup_{s\in J}\beta(H(s)))B_0$$

where $P=\bigcup_{i=1}^n P_i$. This implies that $\beta(H(J))\leqslant 2\varepsilon+\sup_{t\in J}\beta(H(t))$. As the above inequality holds for any $\varepsilon>0$, we get $\beta(H(J))\leqslant\sup\beta(H(t))$, which ends the proof.

Given auy $\eta>0$ let us denote by S_η the set of all functions $u:J\to E$ such that $u(0)=x_0$, $\|u(t)-u(s)\|\leqslant M|t-s|$ for $t, s\in J$ and

$$\sup_{t\in J}\|u(t)-x_0-\int_0^t f(s, u(s))\,ds\|<\eta.$$

In [9] it is shown that S_η is a nonempty and connected subset of $C_w(J, E)$.

Now we prove a Kneser type theorem for the problem (CP). Assume that a nonnegative real-valued function h is non-decreasing on R_+ and $u(t)\equiv 0$ is the unique solution of the integral equation

$$z(t)=\int_0^t h(z(s))\,ds \text{ on } J.$$

Theorem. *If for every subset X of B*

$$\beta(f(J\times X))\leqslant h(\beta(X)), \tag{1}$$

then the set S of all weak solutions of the Cauchy problem (CP) *defined on J is nonempty, compact and connected in $C_w(J, E)$.*

Proof. Let $\tilde{B}$ denote the set of all weakly continuous functions $x:J\to B$. Put

$$F(x)(t)=x_0+\int_0^t f(s, x(s))\,ds \text{ for } t\in J \text{ and } x\in\tilde{B}.$$

It is clear from Lemma 1 that F is a continuous mapping of $\tilde{B}$ into $C_w(J, E)$.

1. First we shall show that the set S is nonempty. For any $n\in N$ we choose $u_n\in S_{1/n}$. Let $H=\{u_n : n\in N\}$.

It follows from the definition of S_η that $\|u_n(t)-u_n(s)\|\leqslant M|t-s|$ for $n=1, 2, \ldots$ and $t, s\in J$. Hence $|\beta(H(t))-\beta(H(s))|\leqslant 2M|t-s|$, which proves the continuity of the function $t\to v(t)=\beta(H(t))$ on J.

For fixed $t\in J$ we divide the interval $[0, t]$ into m parts: $0=t_0<t_1<\cdots<t_m=t$, where $t_i=it/m$, $i=0, 1, \ldots, m$. Let

$$H([t_{i-1}, t_i])=\{u(s): u\in H,\ t_{i-1}\leqslant s\leqslant t_i\}.$$

By Lemma 2 and the continuity of v there is $s_i\in[t_{i-1}, t_i]$ such that

$$\beta(H([t_{i-1}, t_i]))=\sup\{\beta(H(s)): t_{i-1}\leqslant s\leqslant t_i\}=v(s_i). \tag{2}$$

On the other hand, by the mean value theorem [1; Th. V. 10. 4.] we obtain

$$F(u)(t)=x_0+\sum_{i=0}^{m-1}\int_{t_i}^{t_{i+1}} f(s, u(s))\,ds\in x_0+\sum_{i=0}^{m-1}(t_{i+1}-t_i)\,\overline{\operatorname{conv}}\, f(J\times H([t_i, t_{i+1}]))$$

for each $u\in H$. Therefore

$$F(H)(t)\subset x_0+\sum_{i=0}^{m-1}(t_{i+1}-t_i)\,\overline{\operatorname{conv}}\, f(J\times H([t_i, t_{i+1}])).$$

By (1), (2) and the corresponding properties of β it follows that

$$\beta(F(H)(t)) \leqslant \sum_{i=0}^{m-1} (t_{i+1}-t_i)\,\beta(f(J\times H([t_i, t_{i+1}])))$$

$$\leqslant \sum_{i=0}^{m-1} (t_{i+1}-t_i)\,h(\beta(H([t_i, t_{i+1}]))) = \sum_{i=0}^{m-1} (t_{i+1}-t_i)\,h(v(s_i)).$$

But if $m\to\infty$, then $\sum_{i=0}^{m-1} (t_{i+1}-t_i)\,h(v(s_i)) \longrightarrow \int_0^t h(v(s))\,ds$. Thus

$$\beta(F(H)(t)) \leqslant \int_0^t h(v(s))\,ds \quad \text{for } t\in J. \tag{3}$$

Because $\lim_{n\to\infty} \| u_n - F(u_n)\|_c = 0$ and

$$H(t) \subset \{u_n(t) - F(u_n)(t) : u_n \in H\} + F(H)(t),$$

so $\beta(H(t)) \leqslant \beta(F(H)(t))$, and finally, by (3)

$$v(t) \leqslant \int_0^t h(v(s))\,ds \quad \text{for } t\in J.$$

Applying now theorem on differential inequalities (cf. [6]) we get $v(t) = \beta(H(t)) = 0$ for each $t\in J$. Consequently $H(t)$ is conditionally weakly compact. Hence Ascoli's Theorem [7] proves that H is conditionally compact in $C_w(J, E)$. Therefore the sequence (u_n) has a limit point u.

Since $\lim(u_n - F(u_n)) = 0$ and F is continuous, we obtain that $u = F(u)$, i.e. $u\in S$.

2. Now we shall prove that the set S is compact.

Since F is continuous, S is closed in $C_w(J, E)$. As $S = F(S)$, by repeating the argument from 1 we can show that S is conditionally compact in $C_w(J, E)$.

Suppose that the set S is not connected. As S is compact, thus there are nonempty compact sets W_1 and W_2 such that $S = W_1 \cup W_2$ and $W_1 \cap W_2 = \varnothing$, and consequently there are two disjoint open sets U_1, U_2 such that $W_1 \subset U_1$, $W_2 \subset U_2$. Suppose that for every $n\in N$ there exists a $u_n \in V_n \setminus U$, where $V_n = \overline{S}_{1/n}$ and $U = U_1 \cup U_2$. Put $H = \overline{\{u_n : u\in N\}}$. Since $u_n - F(u_n) \to 0$ if $n\to\infty$, by repeating the argument from **1** we see that there exists $u_0 \in H$ such that $u_0 = F(u_0)$, i. e. $u_0 \in S$. Furthermore $S \subset C_w(J, E) \setminus U$, since U is open, so that $u_0 \in S\setminus U$, a contradiction.

Therefore there is $m\in N$ such that $V_m \subset U$. Since $U_1 \cap V_m \neq \varnothing \neq U_2 \cap V_m$, this shows that V_m is not connected, in contradiction with the connectedness of all V_n. Hence S is connected..

R e m a r k. The assumption (1) was used by Evin Cramer, V. Lakshmikantham and A.R. Mitchel in [4] in the proof of existence theorem, and is analogous to the corresponding assumptions for measure of noncompactness (cf. [3], [5]).

REFERENCES

[1] A. Alexiewicz, *Analiza funkcjonalna*, PWN, Warszawa, 1968.

[2] F.S. de Blasi, *On a property of the unit sphere in a Banach space*, Bull. Math. Soc. Sci. Math. R.S. Roumanie, **21** (1977) 259—262.

[3] A. Cellina, *On the existence of solutions of ordinary differential equations in Banach spaces*, Funkcial. Ekvac. **14** (1971), 129—136.

[4] E. Cramer, V. Lakshmikantham, A.R. Mitchell; *On the existence of weak solutions of differential equations in nonreflexive Banach spaces*, Nonlinear Anal. **2**, (1978), 169—177.

]5[K. Goebel, W. Rzymowski, *An existence theorem for the equations* $x' = f(t, x)$ *in Banach space*, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astronom. Phys., **18** (1970), 367—370.

[6] P. Hartman. *Ordinary Differential Equations*, Wiley and Sons, New York, 1964.

[7] J. L. Kelley, *General Topology*, Toronto, New York, London, 1957.

[8] A. Szep, *Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces*, Studia Sci. Math. Hungar. **6** (1971), 197—203.

[9] S. Szufla, *Kneser's Theorem for weak solutions of ordinary differential equations in reflexive Banach spaces*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., **26**(1978), 407—413.

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(Received 21 08 1980)