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SUBOPTIMALITY OF STOCHASTIC SYSTEMS: STRUCTURAL UNCERTAINTIES AND INFORMATION CONSTRAINTS*

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Abstract. Suboptimality characterization of linear feedback with complete and incomplete state information is presented for linear time-varying systems corrupted by input and output noise. The measure of performance degradation is proposed to quantify the loss caused by the use of suboptimal feedback loop in the presence of structural uncertainties and nonclassical information constraints. Necessary and various sufficient conditions for suboptimality are given.

1. Introduction. *A desing methodology* for suboptimal estimation and control in the presence of *model uncertainties* and *nonclassical constraints on control and information patterns* is frequently needed in feedback design of large-scale engineering systems composed of interconnected subsystems. Generalizing the previous deterministic work on performance deterioration and suboptimality of decentralized control of weakly coupled systems, [2, 9 — 11, 13, 14], we have provided *characterizations of suboptimality* of linear feedback for stochastic systems with deterministic initial state and zero terminal cost, [8]. The purpose of the present paper is to extend this characterization to the general *LQG* problem with an arbitrary structural perturbation that affects the input, the output and the interconnections. The degree of suboptimality being defined as an upper bound of the deviation of the performance with respect to the performance of a referent system, the measure of performace degradation with respect to the best possible performance is introduced.

In Section 2., we define the *degree of suboptimality* of a control law with complete state information, and provide the corresponding necessary and various sufficient conditions for the *control to be suboptimal.*

In Section 3., we extend the notion of *degree of suboptimality* to the linear feedback with *incomplete state information.* Conditions that are presented here are a natural generalization to those presented in the previous section.

Notation: with some obvious exceptions, Greek letters denote scalars, lower case italic letters denote vectors, and capital italic letters denote matrices.

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2. Degree of suboptimality: complete state information. Let us consider a set of s interconnected subsystems described by stochastic differential equations

$$(2.1)\qquad dx_i = A_i x_i dt + B_i u_i dt + \sum_{j=1}^{s} A_{ij} x_j dt + \sum_{j=1}^{s} B_{ij} u_j dt + dv_i, \; i = 1, 2, \dots, s,$$

where $x_i(t)$, $u_i(t)$, $v_i(t)$ are n_i, p_i, n_i vectors, respectively, which are the state, input and input-noise of the i-th subsystem such that $n_1 + n_2 + \cdots + n_s = n$, $p_1 + p_2 + \cdots + p_s = p$, $(p \leqslant n)$, and $A_i(t)$, $A_{ij}(t)$, $B_i(t)$, $B_{ij}(t)$ are matrices of appropriate dimensions, whose elements are continuous functions of time on the interval (t_0, t_f); $t_0 \geqslant 0$ is the initial time, and $t_f > t_0$ is the final, terminal time. The input noise in (2.1) is described by an n vector $v(t) = [v_1^T(t), v_2^T(t), \dots, v_s^T(t)]$, which is a Wiener process independent of the inital state $x_0^T = [x_1^T(t_0), x_2^T(t_0), \dots, x_s^T(t_0)]$, with $E(dvdv^T) = R_v dt$, were E denotes mathematical expectation, $R_v(t) = \mathrm{diag}\{R_{v1}(t), R_{v2}(t), \dots, R_{vs}(t)\}$ is the incremental covariance matrix, whose elements are uniformly bounded continuous functions of time, and $E(dv) = 0$. It is also assumed that the Guassian statistics $E(x_0) = m_0$, $cov(x_0, x_0) = R_0$, and $R_v(t)$, for the system (2.1) is given *a priori*.

For convenience, we introduce the notation $A_D = \mathrm{diag}\{A_1, A_2, \dots, A_s\}$, $B_D = \mathrm{diag}\{B_1, B_2, \dots, B_s\}$, $A_C = (A_{ij})$, $B_C = (B_{ij})$, $(i, j = 1, 2, \dots s)$, $A = A_D + A_C$, $B = B_D + B_C$, where the subscripts D and C stand for "decoupled" and "coupled" subsystems. With this notation, the system (2.1) can be represented as

$$(2.2)\qquad dx = A_D x dt + B_D u dt + A_C x dt + B_C u dt + dv,$$

where $x^T(t) = [x_1^T(t), x_2^T(x), \dots, x_s^T(t)]$ and $u^T(t) = [u_1^T(t), u_2^T(t), \dots, u_s^T(t)]$.

We associate with the i-th subsystem in (2.1) a quadratic cost

$$(2.3)\qquad J_i = x_{fi}^T S_{fi} x_{fi} + \int_{t_0}^{t_f} (x_i^T Q_{xi} x_i + u_i^T Q_{ui} u_i)\, dt,$$

and consider the expected value EJ_i of the cost J_i as a measure of system performance. We denote by $x_{fi} = x_i(t_f)$ and $x_{fi}^T S_{fi} x_{fi}$ the terminal (final) state and the terminal cost of i-th subsystem, respectively. The matrices $Q_{xi}(t)$ are symmetric positive semidefinite of dimension $n_i \times n_i$, the matrices $Q_{ui}(t)$ are symmetric positive definite of dimension $p_i \times p_i$ and the matrices S_{fi} are positive semidefinite of dimension $n_i \times n_i$. It is assumed that the blockdiagonal matrices $Q_x = \mathrm{diag}\{Q_{x1}, Q_{x2}, \dots, Q_{xs}\}$, $Q_u = \mathrm{diag}\{Q_{u1}, Q_{u2}, \dots, Q_{us}\}$ and Q_u^{-1} have uniformly bounded continuous elements in time. Hence, the quadratic cost associated to the interconnected system (2.2) is

$$(2.4)\qquad J = x_f^T S_f x_f + \int_{t_0}^{t_f} (x^T Q_x x + u^T Q_u u)\, dt,$$

where $x_f = x(t_f)$, and $S_f = \text{diag}\{S_{f1}(t_f), S_{f2}(t_f), \dots, S_{fs}(t_f)\}$. A way to account for the incomplete knowledge of the interconnection matrices A_C and B_C is to compare the expected cost of the system (2.2) to the expected cost of some referent system. This is performed by assuming some referent pattern of interconnections with respect to which bounded but otherwise unknown perturbations are allowed. In our case perturbed values of the elements of the interconnection matrices A_C, B_C are defined with respect to zero ($A_C \equiv 0$, $B_C \equiv 0$). Another possibility is to determine the referent interconnection matrix by some nonzero values of its elements (extreme, average, or else). The additional interest in the referent pattern of interconnections $A_C \equiv 0$, $B_C \equiv 0$ resides in the fact that the control design for the referent system (2.2) provides in that case *automatically a decentralized control law* which often proves to be a technological and computational benefit.

We denote by

$$u_D(t) = -L_D(t, t_f)x(t), \quad u(t) = -L(t, t_f)x(t), \tag{2.5}$$

control laws that correspond to the referent system (2.2) in which $A_C \equiv 0$, $B_C \equiv 0$, and to an arbitrary system (2.2) with $A_C \not\equiv 0$, $B_C \not\equiv 0$, respectively. The elements of the $p \times n$ matrices $L_D(t, t_f)$ and $L(t, t_f)$ are continuous functions of time t on the interval (t_0, t_f). In analogy to matrices A_C and B_C, we introduce the matrix $L_C = (L_{ij})$, $(i, j = 1, 2, \dots, s)$, such that $L_C(t, t_f) + L_D(t, t_f) = L(t, t_f)$, the dimensions of submatrices $L_{ij}(t, t_f)$ being $p_i \times n_j$. When the disconnected system ($A_C \equiv 0$, $B_C \equiv 0$) is used as reference, we choose $L_D(t, t_f)$ to be a quasidiagonal matrix $L_D(t, t_f) = \text{diag}\{L_1(t, t_f), L_2(t, t_f), \dots, L_s(t, t_f),\}$, where the dimensions of the i-th submatrix are $p_i \times n_i$, $(i = 1, 2, \dots, s)$; *in other words, it is assumed that* the only knowledge available to the i-th referent controller is the state x_i of the i-th sybsystem.

The value of the expected cost EJ^0 of the referent system (2.2) with control $u_D(t)$ and interconnection matrices $A_C(t) \equiv 0$, $B_C(t) \equiv 0$ is generally different from that of EJ^+ which is calculated for some (nonzero) matrices $A_C(t)$, $B_C(t)$ and the control $u(t)$. We say that the system is *weakly coupled with respect to the cost* (2.4), [2], if EJ^+ is bounded from above by $\mu^{-1}EJ^0$, where μ is a positive number.

In the following definition, we assume that the control law $u_D(x, t)$ is optimal when $A_C \equiv 0$, $B_C \equiv 0$ in (2.2), and that the pairs of matrix functions (A_D, B_D), $(A_D, Q_x^{1/2})$ satisfy the conditions of uniform and differential controllability and observability, respectively, [6, 15, 16].

(2.6) Definition. The control law

$$u(t) = -L(t, t_f)x(t) \tag{2.7}$$

is *suboptimal with degree* μ for the system (2.2) and the cost (2.4) if and only if there exists a *positive number* μ such that the inequality

$$E[J^+ \mid x_0] \leqslant \mu^{-1} E[J^0 \mid x_0] \tag{2.8}$$

holds for all initial states $x_0 \neq 0$, their expectations m_0. and all $t_0 \in (0, +\infty)$' $t_f \in (t_0, +\infty)$.

We note that Definition (2.6) is designed to single out a class of nonstationary suboptimal control problems which are in a sense "quasistationary", that is, independent of choice of the time interval (t_0, t_f), provided $t_f - t_0 > 0$. Otherwise, it is possible to define the degree of suboptimality with respect to some specific time interval, e.g. $(0, +\infty)$ according to (5). To stress the difference, we use the term *uniform suboptimality* for the property described in (2.6). We also note that by the use of conditional expectation in (2.8), Definition (2.6) keeps care of the fact that the *a priori* knowledge of the initial state statistics R_0, m_0 is usually unprecise. Thus, the property of suboptimality (2.8) does not depend on the initial state information.

Finally, we observe that the common notion of (uniform) suboptimality is implied by Definition (2.6) when the system in consideration and the referent system have identical open-loop structures, $(A(t) \equiv A_D(t),\ B(t) \equiv B_D(t))$. In order to describe more explicitly the uniform suboptimality property of the control law (2.7), we introduce the matrices $M(t, t_f)$, $N(t, t_f)$, $\hat{A}(t, t_f)$, $\hat{A}_D(t, t_f)$, $W(t, t_f)$, $W_D(t, t_f)$ defined by

$$\frac{d}{dt} M(t, t_f) + \hat{A}^T(t, t_f) M(t, t_f) + M(t, t_f) \hat{A}(t, t_f) + W(t, t_f) = 0,$$

$$M(t_f, t_f) = S_f,$$

$$\hat{A}(t, t_f) = A(t) - B(t) L(t, t_f),$$

$$W(t, t_f) = L^T(t, t_f) Q_u(t) L(t, t_f) + Q_x(t),$$

(2.9)

$$\frac{d}{dt} N(t, t_f) + \hat{A}_D^T(t, t_f) N(t, t_f) + N(t, t_f) \hat{A}_D(t, t_f) + W_D(t, t_f) = 0,$$

$$N(t_f, t_f) = S_f,$$

$$\hat{A}_D(t, t_f) = A_D(t) - B_D L_D(t, t_f),$$

$$W_D(t, t_f) = L_D^T(t, t_f) Q_u(t) L_D(t, t_f) + Q_x(t).$$

We note that our controllability and observability assumptions imply finiteness and positive definiteness of the matrix $N(t_0, t_f)$ for all $t_f \in (0, +\infty)$ and all $t_0 \in [0, t_f)$. Indeed, it can be shown by a slight modification of arguments in the proof of Kalman's stability theorem, [6, pp. 114—116], (see also [3, pp. 67—69], [4, pp. 724—725], and [18, pp. 687—691]), that these assumptions imply positive definiteness of the matrix $N(t_0, t_f)$ for all $t_f > t_0$. Moreover, the normed quadratic form of the matrix $N(t_0, t_f)$ is uniformly bounded from below and from above by positive definite quadratic forms that depend on $t_f - t_0$ only.

In addition, our observability and controllability assumptions assure that the optimal control problem for the disconnected system is meaningful in the limit $t_f \to +\infty$.

We introduce also the nonnegative number

$$\mu_0 = \inf_{t_0,\, t_f} \lambda_M^{-1} [N^{-1}(t_0,\ t_f)\, M(t_0,\ t_f)],$$

where $\inf_{t_0,\, t_f}$ shorthands $\inf_{\substack{t_f \in (0,\, +\infty) \\ t_0 \in [0,\, t_f)}}$. As usual, λ_M in (2.10) is the largest eigenvalue of the indicated matrix. Now we formulate the following

(2.11) Theorem. *The control law* (2.7) *is suboptimal with degree* μ_0 *for the system* (2.2) *and the cost* (2.4) *if and only if the matrix* $M_0 = \lim_{t_f \to +\infty} M(0,\ t_f)$ *is finite.*

Proof. We remark that

$$(2.12)\qquad \begin{aligned} E[J^+ | x_0] &= x_0^T M(t_0,\ t_f)\, x_0 + \alpha(t_0,\ t_f), \\ E[J^0 | x_0] &= x_0^T N(t_0,\ t_f)\, x_0 + \beta(t_0,\ t_f), \end{aligned}$$

where the scalar functions

$$(2.13)\qquad \begin{aligned} \alpha(t_0,\ t_f) &= \int_{t_0}^{t_f} tr\, [M(t,\ t_f)\, R_v(t)]\, dt, \\ \beta(t_0,\ t_f) &= \int_{t_0}^{t_f} tr\, [N(t,\ t_f)\, R_v(t)]\, dt, \end{aligned}$$

are defined for all $t_f \in (0,\ +\infty)$ and all $t_0 \in [0,\ t_f)$. We denote

$$(2.14)\qquad \begin{aligned} \varphi_M(t_0,\ t_f,\ x_0) &= x_0^T M(t,\ t_f)\, x_0, \\ \varphi_N(t_0,\ t_f,\ x_0) &= x_0^T N(t_0,\ t_f)\, x_0, \end{aligned}$$

$$\varphi(t_0,\ t_f,\ x_0) = [\varphi_M(t_0,\ t_f,\ x_0) + \alpha(t_0,\ t_f)] / [\varphi_N(t_0,\ t_f,\ x_0) + \beta(t_0,\ t_f)].$$

We recall that Definition (2.6) assumes:

(a) optimality of the control law $u_D(x,\ t)$ when $A_C \equiv 0$ and $B_C \equiv 0$,

(b) uniform and differential controllability of the pair of matrix functions $(A_D,\ B_D)$,

(c) uniform and differential observability of the pair of matrix functions $(A_D,\ Q_x^{1/2})$.

Under these assumptions, the matrix $N(t_0,\ t_f)$ is positive definite and uniformly bounded for all $t_f \in (0,\ +\infty)$ and all $t_0 \in [0,\ t_f)$ and for al $x_0 \neq 0$

$$0<\varphi_N(t_0,\ t_f,\ x_0)+\beta(t_0,\ t_f),$$

(2.15)
$$\varphi_N(t_0,\ t_f,\ x_0)\leqslant\rho_N\cdot\|x_0\|^2,$$

$$\beta(t_0,\ t_f)<+\infty,$$

for some positive number ρ_N. The function $\varphi(t_0,\ t_f,\ x_0)$ is well defined in this domain. Hence, we may restate the Definition (2.6) in the following way.

The control law $u(t,\ x)$ is *suboptimal with degree* μ (for the appropriate refrent system and cost) if and only if there exists a positive mumber μ such that

$$\mu\cdot\sup_{\substack{t_0,\ t_f\\ x_0\neq 0}}\varphi(t_0,\ t_f,\ x_0)\leqslant 1. \tag{2.16}$$

It may be easily shown that

$$\sup_{\substack{t_0,\ t_f\\ x_0\neq 0}}\varphi(t_0,\ t_f,\ x_0)>0. \tag{2.17}$$

Indeed, it follows from (c) that $Q_x(t)\neq 0$. Therefore $L^T(t,\ t_f)Q_u(t)L(t,\ t_f)+Q_x(t)\neq 0$ and $M(t_0,\ t_f)\neq 0$, that is, $\operatorname{rank}[M(t_0,\ t_f)]\geqslant 1$. Hence, there is a vector $x_0\neq 0$ such that $x_0^T M(t_0,\ t_f)x_0>0$, given t_0, t_f, and therefore $\varphi(t_0,\ t_f,\ x_0)>0$, which proves the statement.

Hence, the inequality (2.16) never allows all nonnegative values of $\mu\in\in[0,\ +\infty)$. It is evident that the control law $u(t,\ x)$ is suboptimal if and only if $\sup\limits_{\substack{t_0,\ t_f\\ x_0\neq 0}}\varphi(t_0,\ t_f,\ x_0)$ exists, that is, if and only if

$$\sup_{\substack{t_0,\ t_f\\ x_0\neq 0}}\varphi(t_0,\ t_f,\ x_0)<+\infty. \tag{2.18}$$

In that case, the control law is suboptimal with degree $\mu\in(0,\ \mu_0]$, where

$$\mu_0=1/\sup_{\substack{t_0,\ t_f\\ x_0\neq 0}}\varphi(t_0,\ t_f,\ x_0) \tag{2.19}$$

is a defined number.

It remains to prove that the right hand sides of (2.19) and (2.10) are equivalent, and that $\|M_0\|<+\infty$ is indeed a necessary and sufficient condition for the inequality (2.18) to be true. The first task is accomplished by the following

L e m m a. *The function* $\varphi(t_0,\ t_f,\ x_0)$ *—defined by* (2.12—14) *under the conditions* (a, b, c) *—satisfies*

$$\sup_{\substack{t_0,\ t_f\\ x_0\neq 0}}\varphi(t_0,\ t_f,\ x_0)=\sup_{t_0,\ t_f}\lambda_M[N^{-1}(t_0,\ t_f)M(t_0,\ t_f)], \tag{2.20}$$

where λ_M *is the largest eigenvalue of the indicated matrix.*

Proof. Let us denote: $\|x_0\|=\rho$, $e_0=x_0\|x_0\|^{-1}$, $\varphi_m(t_0, t_f, x_0)=e_0^T M(t_0, t_f)e_0$, $\varphi_n(t_0, t_f, x_0)=e_0^T N(t_0, t_f))e_0$, and $\varphi(t_0, t_f, e_0, \rho)=[\varphi_m(t_0, t_f, e_0)\cdot\rho^2+\alpha(t_0, t_f)]/[\varphi_n(t_0, t_f, e_0)\cdot\rho^2+\beta(t_0, t_f)]$, where $\|e_0\|=1$, $\varphi_m(t_0, t_f, e_0)\cdot\rho^2=\varphi_M(t_0, t_f, x_0)$, $\varphi_n(t_0, t_f, e_0)\cdot\rho^2=\varphi_N(t_0, t_f, x_0)$, and $\varphi(t_0, t_f, x_0)\equiv\varphi(t_0, t_f, e_0, \rho)$. We observe that

$$\frac{\partial\varphi}{\partial\rho}=2\,\rho\cdot(\varphi_m\cdot\beta-\varphi_n\cdot\alpha)/(\varphi_n\,\rho^2+\beta)^2. \tag{2.21}$$

Accordingly

(i) when $\varphi_m(t_0, t_f, e_0)/\varphi_n(t_0, t_f, e_0)>\alpha(t_0, t_f)/\beta(t_0, t_f)$

$$\frac{\partial\varphi}{\partial\rho}>0 \text{ for all } \rho\in(0, +\infty);$$

(ii) when $\varphi_m(t_0, t_f, e_0)/\varphi_n(t_0, t_f, e_0)\leqslant\alpha(t_0, t_f)/\beta(t_0, t_f)$,

$$\frac{\partial\rho}{\partial\varphi}\leqslant 0 \text{ for all } \rho\in(0, +\infty).$$

Hence, when the inequality (i) is true

$$\sup_{\rho\in(0,+\infty)}\varphi(t_0, t_f, e_0, \rho)=\varphi_m(t_0, t_f, e_0)/\varphi_n(t_0, t_f, e_0); \tag{2.22}$$

when the inequality (ii) holds

$$\sup_{\rho\in(0,+\infty)}\varphi(t_0, t_f, e_0, \rho)=\alpha(t_0, t_f)/\beta(t_0, t_f). \tag{2.23}$$

Thus, we have reduced the task of calculating

$$\sup_{x_0\neq 0}\varphi(t_0, t_f, x_0)=\sup_{\|e_0\|=1}\ \sup_{\rho\in(0,+\infty)}\varphi(t_0, t_f, e_0, \rho) \tag{2.24}$$

to the evaluation of the expression

$$\max[\alpha(t_0, t_f)/\beta(t_0, t_f),\ \sup_{\|e_0\|=1}\varphi_m(t_0, t_f, e_0)/\varphi_n(t_0, t_f, e_0)]. \tag{2.25}$$

Hence,

$$\sup_{\substack{t_0, t_f\\ x_0\neq 0}}\varphi(t_0, t_f, x_0)= \tag{2.26}$$

$$\max[\sup_{t_0, t_f}\alpha(t_0, t_f)/\beta(t_0, t_f),\ \sup_{\substack{t_0, t_f\\ \|e_0\|=1}}\varphi_m(t_0, t_f, e_0)/\varphi_n(t_0, t_f, e_0)].$$

It is well known, [7], that

$$\sup_{\substack{t_0, t_f\\ \|e_0\|=1}}\varphi_m(t_0, t_f, e_0)/\varphi_n(t_0, t_f, e_0)= \tag{2.27}$$

$$\sup_{t_0, t_f}\lambda_M[N^{-1}(t_0, t_f)M(t_0, t_f)].$$

In order to prove that $\sup_{\substack{t_0, t_f \\ x_0 \neq 0}} \varphi(t_0, t_f, x_0)$ reduces to $\sup_{\substack{t_0, t_f \\ \|e_0\|=1}} \varphi_m(t_0, t_f, e_0)/\varphi_n(t_0, t_f, e_0)$ in (2.26), we shall prove that

$$\sup_{t_0, t_f} \lambda_M[N^{-1}(t_0, t_f) M(t_0, t_f)] \geqslant \sup_{t_0, t_f} \alpha(t_0, t_f)/\beta(t_0, t_f) \tag{2.28}$$

whenever there exists at least one value of $t \in [0, +\infty)$ such that $R_\nu(t) \not\equiv 0$; (when $R_\nu(t) \equiv 0$, $\varphi(t_0, t_f, x_0)$ reduces to $\varphi_m(t_0, t_f, e_0)/\varphi_n(t_0, t_f, e_0)$, and our assertion is obvious). Indeed, let us denote $R_\nu^{1/2}(t) = [r_1(t), r_2(t), \ldots, r_n(t)]$, where $r_i^T(t) = [r_{1i}(t), r_{2i}(t), \ldots, r_{ni}(t)]$, $r_{ji}(t) = r_{ij}(t)$, $(i, j = 1, 2, \ldots, n)$, and observe that $tr[M(t, t_f) R_\nu(t)] = tr[R_\nu^{1/2}(t) M(t, t_f) R_\nu^{1/2}(t)]$. It follows that $tr[M(t, t_f) R_\nu(t)] = \sum_{i=1}^{n} r_i^T(t) M(t, t_f) r_i(t)$. As $R_\nu(t) \neq 0$, continuity of $R_\nu(t)$ implies that there is at least one index value i, $(i = 1, 2, \ldots, n)$, such that $r_i(t) \neq 0$ for some $t \in (t_0, t_f)$, $t_f \in (0, +\infty)$, $t_0 \in [0, t_f)$. Then

$$r_i^T(t) M(t, t_f) r_i(t) \leqslant \lambda_M[N^{-1}(t, t_f) M(t, t_f)] \cdot r_i^T(t) N(t, t_f) r_i(t). \tag{2.29}$$

Let I be a subset of the index set $\{1, 2, \ldots, n\}$ such that $r_i(t) \neq 0$ whenever $i \in I$. Then

$$\sum_{i \in I} r_i^T(t) M(t, t_f) r_i(t) \leqslant \lambda_M[N^{-1}(t, t_f) M(t, t_f)] \sum_{i \in I} r_i^T(t) N(t, t_f) r_i(t), \tag{2.30}$$

and

$$\sum_{i \in I} r_i^T(t) (M(t, t_f) r_i(t) = tr[M(t, t_f) R_\nu(t)], \tag{2.31}$$

$$\sum_{i \in I} r_i^T(t) N(t, t_f) r_i(t) = tr[N(t, t_f) R_\nu(t)].$$

Hence

$$tr[M(t, t_f) R_\nu(t)] \leqslant \lambda_M[N^{-1}(t, t_f) M(t, t_f)] \cdot tr[N(t, t_f) R_\nu(t)]. \tag{2.32}$$

As (2.32) holds whenever $R_\nu(t) \neq 0$, it holds for all $t \in (t, t_f)$, $t_f \in (0, +\infty)$ $t_0 \in [0, t_f)$. Therefore

$$\int_{t_0}^{t_f} tr[M(t, t_f) R_\nu(t)]\, dt \leqslant \int_{t_0}^{t_f} \lambda_M[N^{-1}(t, t_f) M(t, t_f)] \cdot tr[N(t, t_f) R_\nu(t)]\, dt, \tag{2.33}$$

holds for all $t_f \in (0, +\infty)$ and all $t_0 \in [0, t_f)$. It is obvious that

$$\int_{t_0}^{t_f} \lambda_M[N^{-1}(t, t_f) M(t, t_f)] \cdot tr[N(t, t_f) R_\nu(t)]\, dt \leqslant \tag{2.34}$$

$$\sup_{t \in (t_0, t_f)} \lambda_M[N^{-1}(t, t_f) M(t, t_f)] \int_{t_0}^{t_f} tr[N(t, t_f) R_\nu(t)]\, dt.$$

Using the fact that

$$\text{(2.35)} \qquad \sup_{t\in(t_0,t_f)} \lambda_M[N^{-1}(t, t_f)M(t, t_f)] \leqslant \sup_{t_0, t_f} \lambda_M[N^{-1}(t, t_f)M(t, t_f)].$$

and (2.34), we deduce from (2.33) that

$$\text{(2.36)} \qquad \alpha(t_0, t_f)/\beta(t_0, t_f) \leqslant \sup_{t_0, t_f} \lambda_M[N^{-1}(t_0, t_f)M(t_0, t_f)]$$

holds for all $t_f \in (0, +\infty)$ and all $t_0 \in [0, t_f)$, which verifies the inequality (2.28).

Using (2.27) and (2.28) in (2.26) we obtain (2.20) and prove the lemma.

In order to complete the proof of the theorem, it remains to verify that $\|M_0\| < +\infty$ is indeed a necessary and sufficient condition for the inequality (2.18) to hold. Let us consider the ratio of quadratic forms

$$\text{(2.37)} \qquad \varphi_m(t_0, t_f, e_0)/\varphi_n(t_0, t_f, e_0) \equiv \psi(t_0, t_f, e_0),$$

where $\|e_0\| = 1$, $t_f \in (t_0, +\infty)$ and $t_0 \in [0, t_f)$. It follows from (2.15) that $\psi(t_0, t_f, e_0)$ is well defined and that

$$\text{(2.38)} \qquad 0 \leqslant \psi(t_0, t_f, e_0) < +\infty$$

in the domain of allowable values of the arguments t_0, t_f, e_0. Therefore

$$\text{(2.39)} \qquad \sup_{\substack{t_0, t_f \\ \|e_0\|=1}} \psi(t_0, t_f, e_0) = +\infty$$

may be attained only on the boundaries of this domain, (which are not included in the domain itself). As $\psi(t_0, t_f, e_0)$ is a rational function of the entries of e_0, and e_0 belongs to a unit sphere, it is not possible to satisfy (2.39) by an appropriate choice of e_0 when $t_f \in (0, +\infty)$ and $t_0 \in [0, t_f)$ are fixed.

There are two boundary cases that remain to be considered: (i) $t_f - t_0 \to 0$ and (ii) $t_f \to +\infty$.

(i) The first case divides in two subcases:

$$\text{(i.a)} \quad e_0 \in \{e_0 : S_f e_0 = 0\} \quad \text{and} \quad \text{(i.b)} \; e_0 \notin \{e_0 : S_f e_0 = 0\}.$$

(i.a) When $e_0 \in \{e_0 : S_f e_0 = 0\}$, $\lim_{t_0 \to t_f} \psi(t_0, t_f, e_0)$ is indefinite, we use L'Hospital's rule

$$\text{(2.40)} \qquad \lim_{t_0 \to t_f} \psi(t_0, t_f, e_0) = \lim_{t_0 \to t_f} e_0^T \frac{d}{dt} M(t_0, t_f) \cdot e_0 / e_0^T \cdot \frac{d}{dt} N(t_0, t_f) \cdot e_0 .$$

By inspection of the relations (2.9), we observe that

$$\text{(2.41)} \quad e_0^T \cdot \left[\frac{d}{dt_0} M(t_0, t_f) - \frac{d}{dt_0} N(t_0, t_f)\right] \cdot e_0 = -2\, e_0^T \cdot \hat{A}^T(t_0, t_f)\, M(t_0, t_f) \cdot e_0$$

$$-e_0^T \cdot L^T(t_0,\ t_f)\, Q_u(t_0)\, L(t_0,\ t_f) \cdot e_0 +$$

$$+ e_0^T \cdot N(t_0,\ t_f)\, B_D(t_0)\, Q_u^{-1}(t_0)\, B_D^T(t_0)\, N(t_0,\ t_f) \cdot e_0,$$

and therefore

$$\lim_{t_0 \to t_f} e_0^T \cdot \left[\frac{d}{dt_0} M(t_0,\ t_f) - \frac{d}{dt_0} N(t_0,\ t_f)\right] e_0 = -e_0^T \cdot L^T(t_f, t_f)\, Q_u(t_f)\, L(t_f, t_f) \cdot e_0. \tag{2.42}$$

It follows from (2.42) that for sufficiently small $t_f - t_0$, the two quadratic forms are nonnegative and satisfy

$$e_0^T \frac{d}{dt_0} M(t_0,\ t_f) \cdot e_0 \leqslant e_0^T \frac{d}{dt_0} N(t_0,\ t_f) \cdot e_0. \tag{2.43}$$

Using the inequality in (2.40), we obtain

$$\lim_{t_0 \to t_f} \psi(t_0,\ t_f,\ e_0) \leqslant 1. \tag{2.44}$$

(i.b) When $e_0 \notin \{e_0 : S_f e_0 = 0\}$, we obtain by straight calculation

$$\lim_{t_0 \to t_f} \psi(t_0,\ t_f,\ e_0) = 1. \tag{2.45}$$

Hence, (2.39) is not satisfied when $t_0 \to t_f$. Continuity of $\psi(t_0,\ t_f,\ e_0)$ with respect to t_0, t_f implies that (2.39) cannot be satisfied for $t_f - t_0 \to 0$ neither.
(ii) In the second case, that is, when $t_f \to +\infty$, we recall that (2.15) implies $\lim\limits_{t_f - t_0 \to +\infty} \varphi_n(t_0,\ t_f,\ e_0) \leqslant \rho_N$. Therefore, by inspection of (2.37) one can verify that the only possible way to satisfy (2.39) is to put

$$\lim_{t_f \to +\infty} \varphi_m(t_0,\ t_f,\ e_0) = +\infty \tag{2.46}$$

for some $t_0 \in [0,\ t_f)$ and some e_0, $\|e_0\| = 1$. In other words, (2.39) holds if and only if (2.47) holds.

Relation (2.46) implies automatically that at least one entry of the matrix $\lim\limits_{t_f \to +\infty} M(t_0,\ t_f)$ for some $t_0 \in [0,\ t_f)$ is not finite. Due to continuity of $M(t_0,\ t_f)$ in t_0, t_f, this is equivalent to the requirement that the matrix $\lim\limits_{t_f \to +\infty} M(t_0,\ t_f)$ is not finite for all $t_0 \in [0,\ t_f)$. Particularly, we can choose $t_0 = 0$ in order to conform to the statement of the theorem. We remark also that, according to (2.20) and (2.27), relation (2.39) reduces to

$$\sup_{\substack{t_0,\ t_f \\ x_0 \neq 0}} \varphi(t_0,\ t_f,\ x_0) = +\infty. \tag{2.47}$$

Hence, (2.47) holds if and only if the matrix M_0 has at least one entry that is not finite. Conversely,

$$\sup_{\substack{t_0, t_f \\ x_0 \neq 0}} \varphi(t_0, t_f, x_0) < +\infty \tag{2.48}$$

if and only if $\|M_0\| < +\infty$. According to (2.13), this statement proves that $\|M_0\| < +\infty$ is indeed a necessary and sufficient condition for suboptimality of the control law $u(t, x)$. The proof of the theorem is complete.

It follows from Definition (2.6) and Theorem (2.11) that when the open-loop system and the cost are specified, μ_0 determines the maximal admissible degree of suboptimality of a particular feedback law. In other words, the degree of suboptimality μ of the feedback law in consideration is allowed to be any positive number not greater than μ_0. The presence of the same input noise on both the referent and the actual system does not affect the maximal degree of suboptimality μ_0 of the appropriate control law. Although the matrix M_0 is identical with the matrix that guarantees a nonzero degree of suboptimality on the infinite time interval $(0, +\infty)$ for time-invariant deterministic systems, [5], μ_0 is not greater than the best degree of suboptimality defined in [5]. Indeed, according to Theorem (2.11), the convergence of the matrix $M(0, t_f)$ with $t_f \to +\infty$ implies the suboptimal property (2.8) on any specific time interval (t_0, t_f), $(t_f > t_0)$, including the time interval $(0, +\infty)$. On the other hand, by inspection of the expression in (2.10), we observe the fact that the multiplier μ that defines the suboptimality property in (2.8) for some time interval (t_0, t_f) of finite length may happen to be less than the appropriate multiplier for the time interval $(0, +\infty)$. From relations (2.19), (2.26) and (2.45) it follows that $\mu_0 \leqslant 1$ when the matrix S_f is positive definite.

We note also that when the control law $u_D(x, t)$ in (2.5) is optimal, the appropriate differential equation in (2.9) is a Riccati equation, although we do not use this fact explicitly.

The formula (2.10) produces the largest value of the degree of suboptimality, but it does not provide an explicit characterization of the effect of interconnections on suboptimality of the control law (2.7). For this purpose we need a condition which involves the interconnection matrix $\hat{A}_C(t, t_f) = \hat{A}(t, t_f) - \hat{A}_D(t, t_f)$. For the positive definite matrix S_f and also for the case $S_f = 0$, we obtain the following

(2.49) Theorem. *The control law* (2.7) *is suboptimal with degree* μ *for the system* (2.2) *and the cost* (2.4) *if the matrix*

$$F(t_0, t_f; \mu) = \hat{A}_C^T(t_0, t_f) N(t_0, t_f) + N(t_0, t_f) \hat{A}_C(t_0, t_f) - W_D(t_0, t_f) + \mu W(t_0, t_f) \tag{2.50}$$

is negative semidefinite for all $t_0 \in [0, +\infty)$ and all $t_f \in (t_0, +\infty)$.

Proof. Myltiplying the matrix $M(t_0, t_f)$ by μ, subtracting from the matrix $N(t_0, t_f)$, and using the equations (2.9) and (2.50), we obtain

$$\frac{d}{dt_0}(\mu M - N) + \hat{A}^T \cdot (\mu M - N) + (\mu M - N) \cdot \hat{A} + F = 0,$$

(2.51)

$$\mu M(t_f, t_f) - N(t_f, t_f) = -(1-\mu) \cdot S_f.$$

By differentiation with respect to t_0, it can be verified that

$$\mu M(t_0, t_f) - N(t_0, t_f) = \int_{t_0}^{t_f} \hat{\Phi}^T(t, t_0) F(t, t_f; \mu) \hat{\Phi}(t, t_0)\, dt$$

$$(2.52) \qquad -(1-\mu)\, \hat{\Phi}^T(t_f, t_0)\, S_f \hat{\Phi}(t_f, t_0),$$

where $\hat{\Phi}(t, t_0)$ is the state transition matrix defined by

$$(2.53) \qquad \frac{d}{dt}\hat{\Phi}(t, t_0) = \hat{A}(t, t_f) \hat{\Phi}(t, t_0),$$

and $\hat{\Phi}(t_0, t_0) = I$, (we note that the state transition matrix depends on t_f, i.e. $\hat{\Phi} \equiv \hat{\Phi}(t, t_0; t_f)$). It follows from (2.12) that

$$\mu E[J^+ | x_0] - E[J^0 | x_0] = x_0^T [\mu M(t_0, t_f) - N_0(t_0, t_f)] x_0$$

$$(2.54) \qquad + \int_{t_0}^{t_f} tr[\mu M(t, t_f) - N(t, t_f)]\, R_v(t)\, dt,$$

which, together with (2.52), proves the theorem. When the matrix $W(t_0, t_f)$ is positive definite for all $t_0 \in [0, +\infty)$, $t_f > t_0$, the largest number μ^* that verifies the condition of Theorem (2.49) is determined by

$$\mu^* = -\sup_{t_0, t_f} \lambda_M \{W^{-1}(t_0, t_f) [\hat{A}_C^T(t_0, t_f) N(t_0, t_f)$$

$$(2.55) \qquad + N(t_0, t_f) \hat{A}_C(t_0, t_f) - W_D(t_0, t_f)]\}.$$

If $\mu^* > 0$, then μ^* is a degree of suboptimality of (2.7), and obviously $\mu^* \leqslant \mu_0$, so that μ^* provides an estimate of μ_0. This remark parallelizes a comment on the appropriate theorem in [5].

3. Degree of suboptimality: incomplete state information. The results obtained for complete state information may be easily extended to the case of incomplete state information. To show this, let us consider the control problem for the system (2.2) with noisy observation of the output

$$(3.1) \quad dx = A_D\, x dt + B_D\, u dt + A_C\, x dt + B_C\, u dt + dv, \quad dy = C_D\, x dt + C_C\, x dt + dw,$$

where $C(t)=C_D(t)+C_C(t)$ is a $q\times n$ matrix with elements which are continuous functions of time. As usual, we assume the referent output structure to be decentralized, that is $C_D=\mathrm{diag}\{C_1, C_2, \dots, C_s\}$ and $C_C=(C_{ij})$, $(i, j=1, 2, \dots, s)$, where $C_i(t)$, $C_{ij}(t)$ are $q_i\times n_i$ and $q_i\times n_j$ matrices respectively. In (3.1) $w^T(t)=[w_1^T(t), w_2^T(t), \dots, w_s^T(t)]$ is a q vector, the observation noise, which is a Wiener process independent of the initial state x_0 and the input noise $v(t)$. The dimension q of the vector w is determined as $q=q_1+q_2+\cdots+q_s$. Furthemore, $Edw=0$, $E(dwdw^T)=R_w dt$, and $R_w(t)=\mathrm{diag}\{R_{w1}(t), R_{w2}(t), \dots, R_{ws}(t)\}$ is assumed to be positive definite on (t_0, t_f) for all $t_f>t_0$. The elements of $R_w(t)$ and $R_w^{-1}(t)$ are assumed to be uniformly bounded and continuous functions of time. To solve the control problem, we must find a control $u(t)$ as a functional of $y(\tau)$, $\tau\in(t_0, t)$, such that $E[J]$ is minimized. If there is no further restriction on the function $y(\tau)$, $\tau\in(t_0, t)$ in $u(t)$, the resulting LQG problem is said to be with classical information pattern, and has a well known optimal solution, (e.g. [1, p. 289]). In decentralized control, the only information available to the i-th controller is the observation $y_i(t)$ where $y^T(t)=[y_1^T(t), y_2^T(t), \dots, y_s^T(t)]$, and

(3.2) $$dy_i=C_i\,x_i\,dt+dw_i.$$

Therefore the subvectors $u_i(t)$, $(i=1, 2, \dots, s)$, are restricted to be functionals of the subvectors $y_i(\tau)$, $\tau\in(t_0, t)$, and the information pattern is a non-classical one, [17]. Nevertheless, when $A_C\equiv 0$, $B_C\equiv 0$, $C_C\equiv 0$, the separation theorem holds, (e.g. [1, ibid.]), and the optimal control law is defined by the following equations

(3.3) $$\begin{aligned} u_D &= -L_D\hat{x}_D,\\ d\hat{x}_D &= (A_D-B_D L_D)\,\hat{x}_D\,dt+K_D\,(dy-C_D\,\hat{x}_D\,dt), \end{aligned}$$

where $\hat{x}_D(t_0)=m_0$ and $K_D(t, t_0)=\mathrm{diag}\{K_1(t, t_0), K_2(t, t_0), \dots K_s(t, t_0)\}$. The elements of the $q\times n$ matrix $K_D(t, t_0)$, with $q_i\times n_i$ submatrices $K_i(t, t_0)$, are continuous functions of time. The optimal error of the state estimation is denoted by $\tilde{x}(t)=x(t)-\hat{x}_D(t)$.

When the referent system is replaced by a system (3.1) in which at least one of the matrices A_C, B_C, C_C is not identically equal to zero, an arbitrary candidate for the centrally optimal feedback is

(3.4) $$\begin{aligned} u_C &= -L\hat{x},\\ d\hat{x} &= \hat{A}_e\,\hat{x}dt+K(dy-C_e\,\hat{x}dt), \end{aligned}$$

where $\hat{x}(t_0)=m_0$ and the matrices $\hat{A}_e(t, t_f)$, $C_e(t)$ are the estimates of the matrices $\hat{A}(t, t_f)$, $C(t)$. If only the subsystems structure is assumed to be known, $(\hat{A}_e\equiv\hat{A}_D$, $C_e\equiv C_D)$, the decentralized feedback control is obtained for $K(t, t_0)\equiv K_D(t, t_0)$, $L(t, t_f)\equiv L_D(t, t_f)$. The error of the state estimation (3.4) is denoted by $\check{x}(t)=x(t)-\hat{x}(t)$.

The *degree of suboptimality of the linear feedback* (3.4) is defined for the *system with incomplete state information* (3.1) *and the cost* (2.4) by Definition (2.6), where (2.2) and (2.7) are replaced by (3.1) and (3.4), respectively.

The difference must be stressed between suboptimality definition for systems with complete and incomplete state information. Whether the expected cost is conditionned upon the initial state x_0 or not has no importance for the design of the control law when the system states are observed directly. When the state information is not completely available, the precise knowledge of x_0 may be used to design a more accurate filter and thus, the optimization of the conditional and unconditional expected costs produces two different feedback loops. It is not realistic to assume the knowledge of x_0 when the states of the system are not observed directly. Therefore, the *a priori* statistics m_0, R_0 is used in the design of the state estimator, (in other words, the referent estimator optimizes the unconditional mean square estimation error). Still, the resulting cost of the closed-loop system depends on the realization of the initial state, with respect to which the suboptimality property of the feedback has to remain invariant. Therefore, we use the conditional expected cost as the performance index in the definition of a suboptimal feedback with incomplete state information, but we understand that the feedback design was already performed at the time of performance evaluation.

In order to characterize the suboptimality of linear feedback (3.4), let us consider the augmented state vector $z^T(t)=[x^T(t),\ \check{x}^T(t)]$ initialized by $z_0^T=[x_0^T,\ x_0^T-m_0^T]$ and the increment of the augmented Wiener process $de(t)=$ $=[dv^T(t),\ dv^T(t)-dw^T(t)K^T(t,\ t_0)]$, where $\operatorname{cov}[(de(t),\ de(t)]=R_e(t)dt$. Introducing the augmented matrices

$$(3.5)\qquad \begin{aligned} R_e(t;\ t_0)&=\begin{bmatrix} R_v(t) & R_v(t)+K(t,\ t_0)R_w(t)K^T(t,\ t_0) \\ R_v(t) & R_v(t) \end{bmatrix},\\ \tilde{A}^T(t;\ t_0,\ t_f)&=\left[\begin{matrix} \hat{A}^T(t,\ t_f) \\ L^T(t,\ t_f)B^T(t) \end{matrix}\right.\\ &\qquad\left.\begin{matrix} \hat{A}^T(t,\ t_f)-\hat{A}_e^T(t,\ t_f)-C^T(t)K^T(t,\ t_0)+C_e^T(t)K^T(t,\ t_0) \\ \hat{A}_e^T(t,\ t_f)+L^T(t,\ t_f)B^T(t)-C_e^T(t)K^T(t,\ t_0) \end{matrix}\right], \end{aligned}$$

we obtain a compact notation of the system (3.1) with the feedback loop (3.4)

$$(3.6)\qquad dz=\tilde{A}zdt+de,$$

where $z(t)=z_0$. In order to describe the referent system when $K\equiv K_D$, $\hat{A}\equiv\hat{A}_D$, $C\equiv C_e$, we denote by $e_D(t)$, $R_D(t;\ t_0)$, $\tilde{A}_D(t;\ t_0,\ t_f)$ the augmented Wiener process $e(t)$, its incremental covariance $R_e(t;\ t_0)$ and the system state matrix $\tilde{A}(t;\ t_0,\ t_f)$. In that case the state equation of the referent system is

$$(3.7)\qquad dz=\tilde{A}_D zdt+de_D,$$

where $z(t_0)=z_0$. We need two more augmented matrices

$$\tilde{Q}(t, t_f)=\begin{bmatrix} Q_x(t)+L^T(t, t_f)Q_u(t)L(t, t_f) & -L^T(t, t_f)Q_u(t)L(t, t_f) \\ -L^T(t, t_f)Q_u(t)L(t, t_f) & L^T(t, t_f)Q_u(t)L(t, t_f) \end{bmatrix},$$

$$\tilde{Q}_f=\begin{bmatrix} S_f & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.8}$$

in order to obtain the same compact notation for the performance index (2.4) of the closed-loop system (3.6)

$$J=z^T(t_f)\tilde{Q}_f z(t_f)+\int_{t_0}^{t_f} z^T(t)\,Q(t, t_f)\,z(t)\,dt. \tag{3.9}$$

We denote by $\tilde{Q}_D(t, t_f)$ the matrix $\tilde{Q}(t, t_f)$ in the referent case $L(t, t_f)\equiv L_D(t, t_f)$. As before, we use the symbols J^0 and J^+ to indicate the costs of the referent system ($K\equiv K_D$, $L\equiv L_D$, $\hat{A}\equiv\hat{A}_D$, $\hat{A}_e\equiv\hat{A}_D$, $C_e\equiv C_D$) and an arbitrary system ($K\not\equiv K_D$, $L\not\equiv L_D$, $\hat{A}\not\equiv\hat{A}_D$, $\hat{A}\not\equiv\hat{A}_D$, $C_e\not\equiv C_D$), respectively.

When the filter gain is restricted to $K\equiv K_D$, the extension of Theorem (2.11) to the case of incomplete state information is straightforward. We introduce the augmented matrices $\tilde{M}(t, t_f; t_0)$ and $\tilde{N}(t, t_f; t_0)$ analogous to $M(t, t_f)$ and $N(t, t_f)$ in (2.9):

$$\frac{d}{dt}\tilde{M}(t, t_f; t_0)+\tilde{A}^T(t; t_0, t_f)\tilde{M}(t, t_f; t_0)+\tilde{M}(t, t_f; t_0)\,\tilde{A}(t; t_0, t_f)+$$
$$+\tilde{Q}(t; t_f)=0, \quad \tilde{M}(t_f, t_f; t_0)=\tilde{Q}_f,$$

$$\frac{d}{dt}\tilde{N}(t, t_f; t_0)+\tilde{A}_D^T(t; t_0, t_f)\tilde{N}(t, t_f; t_0)+\tilde{N}(t, t_f; t_0)\tilde{A}_D(t; t_0, t_f)+$$
$$+\tilde{Q}_D(t; t_f)=0, \quad \tilde{N}(t_f, t_f; t_0)=Q_f. \tag{3.10}$$

To ensure that the control part of the problem is meaningful in the limit $t_f\to+\infty$ for the referent (disconnected) system, we assume, as before, that the pairs of matrix functions (A_D, B_D), $(A_D, Q_x^{1/2})$ are uniformly controllable and observable respectively. To be sure that the estimation part of the problem makes sense in the limit, we must suppose uniform observability and uniform controllability of the pairs of matrix functions (A_D, C_D) and $(A_D, R_v^{1/2})$, respectively. To guarantee positive definiteness and boundedness of the matrix $\tilde{N}$, we must assume in addition: differential controllability of (A_D, B_D), differential observability of $(A_D, Q_x^{1/2})$, and uniform and differential observability of the pair of matrix functions $(A_D-K_D C_D, Q_u^{1/2}L_D)$. The matrix $\tilde{N}(t_0, t_f; t_0)$ being now nonsingular for all $t_0 \in [0, +\infty)$ and all $t_f \in (t_0, +\infty)$, we introduce the nonnegative number

$$(3.11) \qquad \tilde{\mu}_0 = \inf_{t_0, t_f} \lambda_M^{-1} [\tilde{N}^{-1}(t_0, t_f; t_0) \tilde{M}(t_0, t_f; t_0)]$$

in order to state the following

(3.12) Theorem. *The linear feedback* (3.4) *is suboptimal with degree* $\tilde{\mu} \leqslant \tilde{\mu}_0$ *for the system* (3.1) *and the cost* (2.4) *if and only if the matrix* $\tilde{M}_0 = \lim_{t_f \to +\infty} \tilde{M}(0, t_f; 0)$ *is finite.*

Proof. It is obvious that

$$(3.13) \qquad \begin{aligned} E[J^+ | x_0] &= z_0^T \tilde{M}(t_0, t_f; t_0) z_0 + \tilde{\alpha}(t_0, t_f), \\ E[J^0 | x_0] &= z_0^T \tilde{N}(t_0, t_f; t_0) z_0 + \tilde{\beta}(t_0, t_f), \end{aligned}$$

where $z_0^T = [x_0^T, x_0^T - m_0^T]$, and the scalar functions $\tilde{\alpha}(t_0, t_f)$, $\tilde{\beta}(t_0, t_f)$ are defined by

$$(3.14) \qquad \begin{aligned} \tilde{\alpha}(t_0, t_f) &= \int_{t_0}^{t_f} tr[\tilde{M}(t, t_f; t_0) R_e(t; t_0)] dt \\ \tilde{\beta}(t_0, t_f) &= \int_{t_0}^{t_f} tr[\tilde{N}(t, t_f; t_0) R_D(t; t_0)] dt. \end{aligned}$$

According to Definition (2.6), a necessary condition for suboptimality of the linear feedback (3.4) is that the inequality (2.8) holds for all x_0 and all m_0, that is, for all augmented state vectors z_0. Once this fact is recognized, the same arguments as in Theorem (2.11) are used to complete the proof.

As for the case of complete state information, the matrix inversion in (3.11) requires only the inversion of submatrices of the size not greater than $\max n_i \times \max n_i$, $(i = 1, 2, \ldots, s)$, due to the fact that the augmented matrix $\tilde{N}$ of the referent system has zero off diagonal $n \times n$ submatrices.

The role of the initial state expectation m_0 in the suboptimality definition (2.6) needs some comment. In the case of complete state infomation, the requirement for the inequality (2.8) to be verified for all m_0 is redundant. It is not so in the case of incomplete state information. The omission of m_0 in (2.6) should produce a less restrictive version of Theorem (3.12), as it can be easily verified along the lines of the proof. But it seems that this improvement does not compensate the loss of symmetry with complete state information case and the fact that the precise knowledge of m_0 is hardly to be expected in practice. To provide an explicit characterization of the effect of interconnections on suboptimality of the linear feedback (3.4) for the case $S_f = 0$, we state an analogon to Theorem (2.49).

(3.15) Theorem. *The linear feedback* (3.4) *is suboptimal with degree* $\tilde{\mu}$ *for the system* (3.1) *and the cost* (2.4) *if the matrix*

$$\tilde{F}(t,\ t_f;\ \tilde{\mu})=\tilde{A}_C^T(t_0;\ t_0,\ t_f)\tilde{N}(t_0,\ t_f;\ t_0)+ \tag{3.16}$$
$$\tilde{N}(t_0,\ t_f;\ t_0)\tilde{A}_C(t_0;\ t_0,\ t_f)-\tilde{Q}_D(t_0,\ t_f)+\tilde{\mu}\tilde{Q}(t_0,\ t_f)$$

is negative semidefinite for all $t_0\in[0,\ +\infty)$ *and all* $t_f\in(t_0,\ +\infty)$.

The matrix $\tilde{A}_C(t;\ t_0,\ t_f)$ in (3.16) is defined as $\tilde{A}_C(t;\ t_0,\ t_f)=\tilde{A}(t;\ t_0,\ t_f)-\tilde{A}_D(t;\ t_0,\ t_f)$. When $\tilde{Q}(t_0,\ t_f)$ is positive definite for all $t_0\in[0,\ +\infty)$ and all $t_f\in(t_0,\ +\infty)$, the largest number $\tilde{\mu}^*$ that verifies the condition of Theorem (3.15) is determined by

$$\tilde{\mu}^*=-\sup_{t_0,\,t_f}\lambda_M\{\tilde{Q}^{-1}(t_0,\ t_f)[\tilde{A}_C^T(t_0;\ t_0,\ t_f)\tilde{N}(t_0,\ t_f;\ t_0) \tag{3.17}$$
$$+\tilde{N}(t_0,\ t_f;\ t_0)\tilde{A}_C(t_0;\ t_0,\ t_f)-\tilde{Q}_D(t_0,\ t_f)]\}.$$

4. Conclusion. The concept of uniform suboptimality is introduced in order to characterize the linear feedback loop with and without complete information about the state of linear time-varying systems corrupted by input and output noise, when the input, output, state-feedback and state-interconnection structures are perturbed. In order to provide computational tests, necessary and various sufficient conditions for uniform suboptimality of the linear feedback are derived.

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