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A NOTE ON UNITS AND DIVISORS OF ZERO IN GROUP RINGS

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In this note we show $\Sigma \alpha_g g$ is a unit in the group ring RG if and only if there exists $\Sigma \beta_h h$ in RG such that $\Sigma \alpha_g \beta_{g^{-1}} = 1$ and $\Sigma \alpha_g \beta_h$ is nilpotent whenever $gh \neq 1$ where R is a ring such that if $\alpha\beta = 0$ then $\beta\alpha = 0$; $\alpha, \beta \in R$, and where G is a two unique products group. We also show that if R is a ring with no idempotents $\neq 0$, 1 and whose nilpotent elements form an ideal N, then $J(RG) = NG$ where $J(RG)$ is the Jacobson radical of the group ring RG, G an t.u.p. group.

In the last section we give a necessary and sufficient condition for an element $\Sigma \alpha_i s_i$, $\alpha_i \in R$, $s_i \in S$ to be a divisor of zero in the semigroup ring RS where R is a ring such that if $\alpha\beta = 0$ then $\beta\alpha = 0$; $\alpha, \beta \in R$, and where S is a unique product semigroup.

1. Units. A group G is called a two unique products group if given any two nonempty finite subsets A and B of G with $|A| + |B| > 2$, there exist at least two distinct elements x and y of G that have unique representations in the form $x = ab$, $y = cd$ with $a, c \in A$ and $b, d \in B$. All ordered groups are t.u.p. groups (see [4]). In [2] results about the units of the group ring RG, where R is a ring with identity and G an ordered group, were obtained. In this section we extend these results to the group ring RG where G is any t.u.p. group and suitable restrictions on R. Let $U(RG)$ denote the units of RG.

Lemma 1.1 (*cf* [3], *Lemma* 2.7). *Let R be a ring without nonzero nilpotent elements and $p, q \in RG$ where G is a t.u.p group. If $pq = 1$, where $p = \Sigma \alpha_g g$ and $q = \Sigma \beta_h h$, then $\alpha_g \beta_h = 0$ when $gh \neq 1$.*

Proposition 1.2. *Let R be any ring with identity and let G* be *a t.u.p. group. Then the following are equivalent*

(i) $U(RG) = \{\Sigma \alpha_g g \mid$ *there exists* β_g *in* R *with* $\Sigma \alpha_g \beta_{g^{-1}} = 1$ *and*

$\alpha_g \beta_h = 0$ *whenever* $gh \neq 1\}$

(ii) *R has no non-zero nilpotent elements.*

Proof. That (i) implies (ii) follows from the fact that if $\gamma \in R$ is nilpotent, then $1 + \gamma_g$ is a unit in RG. Lemma 1.1 states the converse. $\square$

Lemma 1.3. *Suppose R is a ring such that if $x, y \in R$ and $xy=0$ then $yx=0$. Then the set of nilpotent elements of R forms an ideal.*

Proof. [1], Lemma 2. □

Theorem 1.4. *Suppose that R satisfies the hypothesis of Lemma 1.3 and let G be a t.u.p. group. Then $\Sigma \alpha_g g$ is a unit in RG if and only if there exists $\Sigma \beta_h h$ in RG such that $\Sigma \alpha_g \beta_{g^{-1}} = 1$ and $\alpha_g \beta_h$ is nilpotent whenever $gh \neq 1$.*

Proof. In [2] this theorem is proved for the case where G is an ordered group. The proof of Theorem 1.3 in [2] is solely dependent on the validity of Proposition 1.2 and Lemma 1.3 for RG. Since this is the case, now, the proof of Theorem 1.4 is the same as when G is ordered. □

Corollary 1.5. *Let R be a ring with identity satisfying hypothesis of Lemma 1.3 with no idempotents $\neq 0$, 1. If G is a t.u.p. group, then $\Sigma \alpha_g g$ is a unit in RG if and only if for some g, α_g is a unit and all other α_g' s are nilpotent.*

Proof. The proof is the same as that of Corollary 1.4. in [2]. □

Corollary 1.6 (*cf.* [3], *Theorem* 2.1). *Let R be a ring with no nilpotent elements $\neq 0$ and no idempotents $\neq 0$, 1. Then the only units in RG are of the form ug where u is a unit of R and g is in G.*

Proof. Since R has no nilpotent elements $\neq 0$ the hypothesis of Lemma 1.3 is satisfied. The result now follows from Corollary 1.5. □

2. Applications. Let $J(RG)$ denote the Jacobson radical of R.

Propostion 2.1. *Suppose R is a ring with no idempotents $\neq 0$, 1 and whose nilpotent elements form an ideal N. Then $J(RG) = NG$ where G is an t.u.p. group*

Proof. This Proposition follows Corollary 1.5 as Proposition 2.1 does in [2]. □

Proposition 2.2. *Let R and S be local rings with no non-zero nilpotent elements. Let G be a t.u.p. group. If $\sigma : RG \to SG$ is a homomorphism then $\sigma(R) \subseteq S$.*

Proof. Again the proof is based on Corollary 1.5 in a similar way as that of Proposition 2.2 in [2]. □

Corollary 2.3. *Let R, S be local rings with 1, and with no non-zero nilpotent elements. Let G be a t.u.p. group. If $\sigma : RG \to SG$ is an isomorphism, then $\sigma(R) = S$.*

3. Divisors of zero in certain semigroup rings. A semigroup S is called a unique product semigroup if, when A and B are non-empty finite subsets of

S, then there always exists at least one $x \in S$ which has a unique representation in the form $x = ab$ with $a \in A$ and $b \in B$. Clearly a *t.u.p.* group is an u.p. group.

Let S be a unique product semigroup and let R be a ring such that if $\alpha, \beta \in R$ and $\alpha\beta = 0$ then $\beta\alpha = 0$.

Lemma 3.1. *If* $a = \sum_{i=1}^{m} \alpha_i s_i$, $\alpha_i \in R$, $s_i \in S$, *and* $b = \sum_{j=1}^{n} \beta_j t_j$, $\beta_j \in R$, $t_j \in S$ *are two non-zero elements of the semigroup ring* RS *such that* $ab = 0$, n *being chosen as small as possible and compatible with* $ab = 0$, *then* $n = 1$.

Proof. If $n = 1$ there is nothing to prove. Suppose $n > 1$. For some p and q, $1 \leqslant p \leqslant m$, $1 \leqslant q \leqslant n$, we have that $s_p t_q \neq s_i t_j$ for $i \neq p$ or $q \neq j$. Since $ab = 0$, it follows that $\alpha_p \beta_q = 0$. Without any loss in generality, we may assume $p = m$ and $q = n$. By assumption $\beta_n \alpha_m = 0$.

Now $a(b\alpha_m) = (ab)\alpha_m = 0$, where $b\alpha_m = \beta_1 \alpha_m t_1 + \beta_2 \alpha_m t_2 + \cdots + \beta_{n-1} \alpha_m t_{n-1}$. By choice of b we must have $b\alpha_m = 0$. Thus $\beta_j \alpha_m = \alpha_m \beta_j = 0$, $j = 1, 2, \ldots, n$. Suppose that, after a suitable re-arrangement of terms, $\alpha_i \beta_j = 0$, $i = d+1, \ldots, m$; $j = 1, 2, \ldots, n$, and that for each i such that $1 \leqslant i \leqslant d$ we have $\alpha_i \beta_j \neq 0$ for some j. Then

$$(\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_d s_d) \cdot (\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n)$$
$$= (\alpha_1 s_1 + \cdots + \alpha_d s_d + \alpha_{d+1} s_{d+1} + \cdots + \alpha_m s_m) \cdot (\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n)$$
$$= ab = 0.$$

From the unique product property of S we may infer $\alpha_p \beta_q = 0$ for some p and q, $1 \leqslant p \leqslant d$, $1 \leqslant q \leqslant n$, and again, without any loss in generality, we assume $p = d$ and $q = n$. Hence $\alpha_d \beta_n = 0$, and consequently $a(b\alpha_d) = (ab)\alpha_d = 0$, where

$$b\alpha_d = \beta_1 \alpha_d t_1 + \beta_2 \alpha_d t_2 + \cdots + \beta_{n-1} \alpha_d t_{n-1} \neq 0.$$

This contradicts the choice of b. Hence $n = 1$. $\square$

Corollary 3.2. *If* $0 \neq p \in RS$ *is a divisor of zero, then there exists a non-zero element* $r \in R$ *such that* $pr = 0$.

Remark. The class of rings for which the condition $\alpha\beta = 0$ implies $\beta\alpha = 0$ holds, includes the class of all rings without non-zero nilpotent elements.

From this remark, it follows that Corollary 3.2 is an extension of [3], Theorem 2.3.

Theorem 3.3. *The element* $a = \sum_{i=1}^{m} \alpha_i s_i$ *of* RS, $\alpha_i \in R$, $s_i \in S$, *is a zero divisor if and only if the ideal* $(0 : A) \neq (0)$ *where* A *is the ideal* $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ *in* R.

Proof. If a is a divisor of zero in RS, then, by Corollary 3.2, an element $\beta \in R$, $\beta \neq 0$, exists such that $\alpha_i \beta = \beta\alpha_i = 0$, $i = 1, \ldots, m$. Let $r \in A$. Then

$$r = \sum_{i=1}^{m} \Big[\sum_{j=1}^{n_i} t_{ij} \alpha_i r_{ij} + t_i \alpha_i + \alpha_i r_i + c_i \alpha_i\Big].$$

with t_{ij}, r_{ij}, t_i, $r_i \in R$ and c_i an integer. Since $(t_{ij}\alpha_i)\beta = 0$ it follows from our assumption on R that $\beta(t_{ij}\alpha_i) = 0$. Similarly, $\beta t_i \alpha_i = 0$. Hence $\beta r = r\beta = 0$, and consequently $(0:A) \neq (0)$.

Conversely, if $(0:A) \neq (0)$ there exists an element $\beta \in R$, $\beta \neq 0$, such that $\alpha_i \beta = \beta\alpha_i = 0$, $i = 1, \ldots, m$. This implies that $(\sum_{i=1}^{m} \alpha_i s_i) \cdot \beta s_k = 0$ for any $s_k \in S$. □

Corollary 3.4. *If R is a commutative Noetherian ring and S a unique product semigroup, then $\sum_{i=1}^{m} \alpha_i s_i$, $\alpha_i \in R$, $s_i \in S$, is a zero divisor in RS if and only if $A = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a zero divisor ideal of R.*

Proof. We need only remark that if A is a zero divisor ideal in R then it is contained in a maximal zero divisor ideal in R and hence $(0:A) \neq (0)$. ([5], Corollary 1, p. 215.) □

The following example shows that R need not be Noetherian. Let V be a rank 1 non-discrete valuation ring and let I be any non-zero principal ideal. The ring $R = V/I$ is not Noetherian. Let $I \subseteq J$ and J be finitely generated. Then $(0:\bar{J}) \neq (0)$ in R and $\bar{J}$ is a zero divisor ideal of R.

Remark. An immediate consequence of the above theorem is that if R is an entire ring (i.e. a ring without non-zero zero divisors) and S is a unique product semigroup, then RS is an entire ring (cf. [4], p. 111).

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