A NOTE ON UNITS AND DIVISORS OF ZERO IN GROUP RINGS

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In this note we show $\Sigma \alpha \cdot g$ is a unit in the group ring $RG$ if and only if there exists $\Sigma \beta \cdot h$ in $RG$ such that $\Sigma \alpha \cdot \beta = 1$ and $\Sigma \alpha \cdot \beta = 0$ is nilpotent whenever $gh \neq 1$ where $R$ is a ring such that if $\alpha \beta = 0$ then $\beta \alpha = 0$; $\alpha, \beta \in R$, and where $G$ is a two unique products group. We also show that if $R$ is a ring with no idempotents $\neq 0, 1$ and whose nilpotent elements form an ideal $N$, then $J(RG) = NG$ where $J(RG)$ is the Jacobson radical of the group ring $RG$, $G$ an t.u.p. group.

In the last section we give a necessary and sufficient condition for an element $\Sigma \alpha \cdot s_{i}, \alpha \in R, s_{i} \in S$ to be a divisor of zero in the semigroup ring $RS$ where $R$ is a ring such that if $\alpha \beta = 0$ then $\beta \alpha = 0$; $\alpha, \beta \in R,$ and where $S$ is a unique product semigroup.

1. Units. A group $G$ is called a two unique products group if given any two nonempty finite subsets $A$ and $B$ of $G$ with $|A| + |B| > 2$, there exist at least two distinct elements $x$ and $y$ of $G$ that have unique representations in the form $x = ab, y = cd$ with $a, c \in A$ and $b, d \in B$. All ordered groups are t.u.p. groups (see [4]). In [2] results about the units of the group ring $RG$, where $R$ is a ring with identity and $G$ an ordered group, were obtained. In this section we extend these results to the group ring $RG$ where $G$ is any t.u.p. group and suitable restrictions on $R$. Let $U(RG)$ denote the units of $RG$.

Lemma 1.1 (cf [3], Lemma 2.7). Let $R$ be a ring without nonzero nilpotent elements and $p, q \in RG$ where $G$ is a t.u.p. group. If $pq = 1$, where $p = \Sigma \alpha \cdot g$ and $q = \Sigma \beta \cdot h$, then $\alpha \cdot \beta = 0$ when $gh \neq 1$.

Proposition 1.2. Let $R$ be any ring with identity and let $G$ be a t.u.p. group. Then the following are equivalent:

(i) $U(RG) = \{ \Sigma \alpha \cdot g | \text{there exists } \beta \cdot h \text{ in } R \text{ with } \Sigma \alpha \cdot \beta = 1 \text{ and } \alpha \cdot \beta = 0 \text{ whenever } gh \neq 1 \}$

(ii) $R$ has no non-zero nilpotent elements.

Proof. That (i) implies (ii) follows from the fact that if $\gamma \in R$ is nilpotent, then $1 + \gamma \cdot g$ is a unit in $RG$. Lemma 1.1 states the converse. □
Lemma 1.3. Suppose $R$ is a ring such that if $x, y \in R$ and $xy = 0$ then $yx = 0$. Then the set of nilpotent elements of $R$ forms an ideal.

Proof. [1], Lemma 2. □

Theorem 1.4. Suppose that $R$ satisfies the hypothesis of Lemma 1.3 and let $G$ be a t.u.p. group. Then $\Sigma a_g g$ is a unit in $RG$ if and only if there exists $\Sigma \beta_h h$ in $RG$ such that $\Sigma a_g \beta_g^{-1} = 1$ and $a_g \beta_h$ is nilpotent whenever $gh \neq 1$.

Proof. In [2] this theorem is proved for the case where $G$ is an ordered group. The proof of Theorem 1.3 in [2] is solely dependent on the validity of Proposition 1.2 and Lemma 1.3 for $RG$. Since this is the case, now, the proof of Theorem 1.4 is the same as when $G$ is ordered. □

Corollary 1.5. Let $R$ be a ring with identity satisfying hypothesis of Lemma 1.3 with no idempotents $\neq 0$, 1. If $G$ is a t.u.p. group, then $\Sigma a_g g$ is a unit in $RG$ if and only if for some $g$, $a_g$ is a unit and all other $a_g$'s are nilpotent.

Proof. The proof is the same as that of Corollary 1.4. in [2]. □

Corollary 1.6 (cf. [3], Theorem 2.1). Let $R$ be a ring with no nilpotent elements $\neq 0$ and no idempotents $\neq 0$, 1. Then the only units in $RG$ are of the form $ug$ where $u$ is a unit of $R$ and $g$ is in $G$.

Proof. Since $R$ has no nilpotent elements $\neq 0$ the hypothesis of Lemma 1.3 is satisfied. The result now follows from Corollary 1.5. □

2. Applications. Let $J(RG)$ denote the Jacobson radical of $R$.

Proposition 2.1. Suppose $R$ is a ring with no idempotents $\neq 0$, 1 and whose nilpotent elements form an ideal $N$. Then $J(RG) = NG$ where $G$ is an t.u.p. group.

Proof. This Proposition follows Corollary 1.5 as Proposition 2.1 does in [2]. □

Proposition 2.2. Let $R$ and $S$ be local rings with no non-zero nilpotent elements. Let $G$ be a t.u.p. group. If $\sigma: RG \to SG$ is a homomorphism then $\sigma(R) \subseteq S$.

Proof. Again the proof is based on Corollary 1.5 in a similar way as that of Proposition 2.2 in [2]. □

Corollary 2.3. Let $R, S$ be local rings with 1, and with no non-zero nilpotent elements. Let $G$ be a t.u.p. group. If $\sigma: RG \to SG$ is an isomorphism, then $\sigma(R) = S$.

3. Divisors of zero in certain semigroup rings. A semigroup $S$ is called a unique product semigroup if, when $A$ and $B$ are non-empty finite subsets of
$S$, then there always exists at least one $x \in S$ which has a unique representation in the form $x = ab$ with $a \in A$ and $b \in B$. Clearly a t.u.p. group is an u.p. group.

Let $S$ be a unique product semigroup and let $R$ be a ring such that if $\alpha, \beta \in R$ and $\alpha \beta = 0$ then $\beta \alpha = 0$.

Lemma 3.1. If $a = \sum_{i=1}^{m} \alpha_i s_i, \alpha_i \in R, s_i \in S$, and $b = \sum_{j=1}^{n} \beta_j t_j, \beta_j \in R, t_j \in S$ are two non-zero elements of the semigroup ring $RS$ such that $ab = 0$, $n$ being chosen as small as possible and compatible with $ab = 0$, then $n = 1$.

Proof. If $n = 1$ there is nothing to prove. Suppose $n > 1$. For some $p$ and $q$, $1 \leq p \leq m, 1 \leq q \leq n$, we have that $s_p t_q \neq s_i t_j$ for $i \neq p$ or $q \neq j$. Since $ab = 0$, it follows that $\alpha_p \beta_q = 0$. Without any loss in generality, we may assume $p = m$ and $q = n$. By assumption $\beta_n \alpha_m = 0$.

Now $a(b \alpha_m) = (ab) \alpha_m = 0$, where $b \alpha_m = \beta_1 \alpha_m t_1 + \beta_2 \alpha_m t_2 + \cdots + \beta_{n-1} \alpha_m t_{n-1}$. By choice of $b$ we must have $b \alpha_m = 0$. Thus $\beta_j \alpha_m = \alpha_m \beta_j = 0, j = 1, 2, \ldots, n$. Suppose that, after a suitable re-arrangement of terms, $\alpha_i \beta_j = 0, i = d + 1, \ldots, m; j = 1, 2, \ldots, n$, and that for each $i$ such that $1 \leq i \leq d$ we have $\alpha_i \beta_j \neq 0$ for some $j$. Then

\[
(\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_d s_d) \cdot (\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n)
\]

\[
= (\alpha_1 s_1 + \cdots + \alpha_d s_d + \alpha_{d+1} s_{d+1} + \cdots + \alpha_m s_m) \cdot (\beta_1 t_1 + \beta_2 t_2 + \cdots + \beta_n t_n)
\]

\[
= ab = 0.
\]

From the unique product property of $S$ we may infer $\alpha_p \beta_q = 0$ for some $p$ and $q$, $1 \leq p \leq d, 1 \leq q \leq n$, and again, without any loss in generality, we assume $p = d$ and $q = n$. Hence $\alpha_d \beta_n = 0$, and consequently $a(b \alpha_d) = (ab) \alpha_d = 0$, where

\[
b \alpha_d = \beta_1 \alpha_d t_1 + \beta_2 \alpha_d t_2 + \cdots + \beta_{n-1} \alpha_d t_{n-1} \neq 0.
\]

This contradicts the choice of $b$. Hence $n = 1$. $\square$

Corollary 3.2. If $0 \neq p \in RS$ is a divisor of zero, then there exists a non-zero element $r \in R$ such that $pr = 0$.

Remark. The class of rings for which the condition $\alpha \beta = 0$ implies $\beta \alpha = 0$ holds, includes the class of all rings without non-zero nilpotent elements.

From this remark, it follows that Corollary 3.2 is an extension of [3], Theorem 2.3.

Theorem 3.3. The element $a = \sum_{i=1}^{m} \alpha_i s_i$ of $RS, \alpha_i \in R, s_i \in S$, is a zero divisor if and only if the ideal $(0 : A) \neq (0)$ where $A$ is the ideal $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ in $R$.

Proof. If $a$ is a divisor of zero in $RS$, then, by Corollary 3.2, an element $\beta \in R, \beta \neq 0$, exists such that $\alpha_i \beta = \beta \alpha_i = 0, i = 1, \ldots, m$. Let $r \in A$. Then

\[
r = \sum_{i=1}^{m} \left[ \sum_{j=1}^{n} t_j \alpha_i r_{ij} + t_i \alpha_i + \alpha_i r_i + c_i \alpha_i \right].
\]
with \( t_{ij}, r_{ij}, t, r \in R \) and \( c_i \) an integer. Since \( (t_{ij} \alpha_i) \beta = 0 \) it follows from our assumption on \( R \) that \( \beta (t_{ij} \alpha_i) = 0 \). Similarly, \( \beta t \alpha_i = 0 \). Hence \( \beta r = r \beta = 0 \), and consequently \( (0 : A) \neq (0) \).

Conversely, if \( (0 : A) \neq (0) \) there exists an element \( \beta \in R, \beta \neq 0 \), such that \( \alpha_i \beta = \beta \alpha_i = 0 \), \( i = 1, \ldots, m \). This implies that \( \sum_{i=1}^{m} \alpha_i s_i \cdot \beta s_k = 0 \) for any \( s_k \in S \).

**Corollary 3.4.** If \( R \) is a commutative Noetherian ring and \( S \) a unique product semigroup, then \( \sum_{i=1}^{m} \alpha_i s_i, \alpha_i \in R, s_i \in S \), is a zero divisor in \( RS \) if and only if \( A = (\alpha_1, \alpha_2, \ldots, \alpha_m) \) is a zero divisor ideal of \( R \).

**Proof.** We need only remark that if \( A \) is a zero divisor ideal in \( R \) then it is contained in a maximal zero divisor ideal in \( R \) and hence \( (0 : A) \neq (0) \). ([5], Corollary 1, p. 215.)

The following example shows that \( R \) need not be Noetherian. Let \( V \) be a rank 1 non-discrete valuation ring and let \( I \) be any non-zero principal ideal. The ring \( R = V/I \) is not Noetherian. Let \( I \subseteq J \) and \( J \) be finitely generated. Then \( (0 : J) \neq (0) \) in \( R \) and \( J \) is a zero divisor ideal of \( R \).

**Remark.** An immediate consequence of the above theorem is that if \( R \) is an entire ring (i.e. a ring without non-zero zero divisors) and \( S \) is a unique product semigroup, then \( RS \) is an entire ring (cf. [4], p. 111).

**REFERENCES**


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