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GENERALIZATION OF AN INEQUALITY OF G. PÓLYA CONCERNING THE EIGENFREQUENCES OF VIBRATING BODIES

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1. If Ω is a domain of R^N (the Euclidean N-space, $N \geqslant 1$) whose closure is C^2 -diffeomorphic with $\overline{B^N}$ (the closure of the open unit ball B^N in R^N), then it is well known ([1], [2]) that the eigenvalues of the Dirichlet problem

$$\Delta u + \Lambda^2 u(x) = 0 \quad (x \in \Omega) \qquad (1)$$
$$u|_{\partial\Omega} = 0$$

form a positive monotone increasing sequence $0 < \Lambda_1(\Omega) \Delta \leq \Lambda_2(\Omega) \leq \ldots$ and they can be given by

$$\Lambda_j^2(\Omega) = \inf_{L \in M_j} \sup_{f \in L} \frac{\int_\Omega |\operatorname{grad} f(y)|^2 dy}{\|f\|_{L^2(\Omega)}} \qquad (2)$$

where M_j is standing for the family of the j-dimensional subspaces of $C_0^\infty(\Omega)$. The values $\Lambda_j(\Omega) = +\sqrt{\Lambda_j^2(\Omega)}$ are interpreted physically as the j-th eigenfreqency of the homogeneous $\overline{\Omega}$ sharped and at the boundary fixed vibrating body.

For $N=2$ Payne and Weinberger ([10]) proved that

$$\sup_\Omega \left[\Lambda_1(\Omega) \frac{\text{area}\,\Omega}{\text{length}\,\partial\Omega}\right] = J$$

where the supremum is taken over all possible Ω-s. On the other hand developing an idea of E. Makai [3], G. Pólya [4] obtained the estimate

$$\sup_{\Omega \in \mathcal{C}} \left[\Lambda_1(\Omega) \frac{\text{area}\,\Omega}{\text{length}\,\partial\Omega}\right] = \frac{\pi}{2}, \qquad (3)$$

where $\mathcal{C}$ denotes the family of all open convex subsets of R^2 It is not hard to see that Pólya's method applies also for N-dimensional convex bodies. and yields[1])

$$C_1(\mathcal{C}) = \sup_{\Omega \in \mathcal{C}} \left[\Lambda_1(\Omega) \frac{\text{vol}_N \Omega}{\text{vol}_{N-1} \partial\Omega}\right] = \frac{\pi}{2}. \qquad (3')$$

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[1]) vol_k denotes the k-dimensional Hausdorff measure.

However

$$\sup_{\Omega}\left[\Lambda_1(\Omega)\frac{\mathrm{vol}_N\,\Omega}{\mathrm{vol}_{N-1}\,\partial\,\Omega}\right]=\infty$$

up to three dimension as it is pointed out in [13] by E. Makai (if Ω ranges over all R^N-domains described at the beginning of this paper).

Therefore it may have some interest to look for large classes $\mathcal{K}$ of R^N-domains such that

$$\sup_{\Omega\in\mathcal{K}}\left[\Lambda_1(\Omega)\frac{\mathrm{vol}_N\,\Omega}{\mathrm{vol}_{N-1}\partial\,\Omega}\right]<\infty$$

Here we shall prove the following

Theorem *Denote by $\mathcal{K}$ the set of those domains $\Omega\subset R^N$ $(N\geq 3)$ for which there exists a C^2-diffeomorphism $T\colon B^N\leftrightarrow\Omega$ and for which the Minkowski curvature with recpect to the outward from Ω directed normal vector of $\partial\Omega$ is non negative at any point of $\partial\Omega$. Then*

$$c_1(\mathcal{K})=\sup_{\Omega\in\mathcal{K}}\left[\Lambda_1(\Omega)\frac{\mathrm{vol}_N\,\Omega}{\mathrm{vol}_{N-1}\,\partial\,\Omega}\right]=\frac{\pi}{2} \tag{4}$$

Proof. First we prove the inequality $c_1(\mathcal{K})\leqq\pi/2$. It is based on two observations.

Lemma 1. *For any open subset Ω of R^N $(N\geqq 1)$ such that $\sup_{\rho>0}\mathrm{vol}_{N-1}\,\partial\Omega_{-\rho}<\infty(\Omega_{-\rho}\stackrel{\text{def}}{=}\{x\in\Omega:\ dist\ (x,\partial\,\Omega)>\rho\})$ we have*

$$\Lambda_1(\Omega)\frac{\mathrm{vol}_N\,\Omega}{\mathrm{vol}_{N-1}\,\partial\,\Omega}\leqq\frac{\pi}{2}. \tag{5}$$

Lemma 2. *For each $\Omega\in\mathcal{K}$ the function $\rho\mapsto\mathrm{vol}_{N-1}\,\partial\,\Omega_{-\rho}$ is monotone decreasing in* $[0,\infty)$ *(and hence* $\mathrm{vol}_{N-1}\,\partial\Omega_{-\rho}\leqq\lim_{\rho\to 0}\mathrm{vol}_{N-1}\,\partial\,\Omega_{-\rho}=\mathrm{vol}_{N-1}\,\partial\,\Omega$).

To prove the converse inequality (i. e. $c_1(\mathcal{K})\geqq\pi/2$) we need only to remark that, by a theorom of R. Courant [11], the function

$$R_1(\Omega)=\Lambda_1(\Omega)\frac{\mathrm{vol}_N\,\Omega}{\mathrm{vol}_{N-1}\partial\,\Omega}$$

is continuous in the topology generated by the Hausdorff metric on $\mathcal{C}$, whence

$$C_1(\mathcal{K})\geqq\sup_{\Omega\in\mathcal{C}\cap\mathcal{K}}R_1(\Omega)=\sup R_1(\Omega)=\frac{\pi}{2}$$

2. Proof of Lemma 1. This statement is essentially proved by G. Pólya in [4]. For the sake of completeness we sketch it here concisely. Set $\rho_0=\sup\{\rho>0:\Omega_{-\rho}\neq 0\}$, $s=\sup_{\rho>0}\mathrm{vol}_{N-1}\,\Omega_{-\rho}$, $l=\mathrm{vol}_N\,\Omega/s$ and let $\xi\colon(0,\rho_0)\to(0,l)$ be the function defined by

$$\xi(\rho)=(\mathrm{vol}_N\Omega-\mathrm{vol}_N\,\Omega_{-\rho})/s$$

and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, respectively. By (2) we obtain

$$\Lambda_1^2(\Omega) = \inf_{f \in C_0^\infty(\overline{\Omega})} \frac{\int_\Omega |\operatorname{grad} f|^2}{\|f\|_{L^2(\Omega)}} \leqslant \inf_{\varphi \in C_0^1(0,\infty)} \frac{\int_\Omega |\operatorname{grad}(\varphi\circ\xi\circ\delta)|^2}{\int_\Omega |\varphi\circ\xi\circ\delta|^2}.$$

Applying [7 Theorem 3.2.12, p. 249] to perform substitutions in the right side of the above inequality, we see that

$$\Lambda_1^2(\Omega) \le \inf_{\varphi \in C_0^1(0,\infty)} \frac{\int_0^{\rho_0} [\varphi'(\xi(\rho))]^2 \cdot [\xi'(\rho)]^2 \operatorname{vol}_{N-1} \partial\Omega_{-\rho}\, d\rho}{\int_0^{\rho_0} [\varphi(\xi(\rho))]^2 \operatorname{vol}_{N-1} \partial\Omega_{-\rho}\, d\rho} =$$

$$= \inf_{\varphi \in C_0^1(0,\infty)} \frac{\int_0^l [\varphi'(\xi)]^2 \cdot [\operatorname{vol}_{N-1} \partial\Omega_{-\rho}]^2\, d\xi}{\int_0^l \varphi(\xi)^2\, d\xi \cdot s} \le \inf_{\varphi \in C_0^1(0,\infty)} \frac{\int_0^l (\varphi')^2}{\int_0^l \varphi^2}.$$

Since $\xi'(\rho) = \operatorname{vol}_{N-1} \partial\Omega_{-\rho}/s$ for almost all $\rho \in (0, \rho_0)$ (cf. [7,3.2.34, p. 271]). Observe that the last term here is the principal eigenfrequency of a vibrating chord of length l fixed only at one of its endpoints. Therefore, as it is well known (cf. [1, p. 71]) the value of the right side is equal to $(\pi/2l)^2$. which completes the proof of Lemma 1.

3. Proof of Lemma 2. Since Ω is C^2-diffeomorphic to $\overline{B^N}$, there exist a point $p \in \partial\Omega$ and a C^2-mapping $F\colon R^{N-1} \to R^N$ having a non-degenerate derivative tensor[2]) everywhere on R^{N-1} such that F constitutes a one to one correspondence between R^{N-1} and $(\partial\Omega)$-$\{p\}$. Let $n(x)$ denote the outward from Ω directed normal vector (with unit length) of $\partial\Omega$ at the point $F(x)$. Introducing the mapping $F^*\colon R^{N-1} \times R \to R^N$ defined by

$$F^*(x, \xi) \equiv F(x) + \xi \cdot n(x) \tag{6}$$

and the function on R^{N-1}

$$h(x) \equiv \sup\{\rho \geqslant 0 : \operatorname{dist}(F^*(x, -\rho), \partial\Omega) = \rho\}, \tag{7}$$

it is easy to observe that F^* constitutes a one to one correspondence between the sets $D \equiv \{(x, -\rho) \in R^N : 0 < \rho < h(x),\ x \in R^{N-1}\}$ and $F^*(D)$.

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[2]) i.e. the rank of the matrix $\dfrac{\partial F}{\partial x} = \left[\dfrac{\partial F_i}{\partial x_j}\right]_{i=1,\, j=1}^{N \quad N-1}$ is equal to $N-1$ at every point of R^{N-1}.

As it is well known, the mapping $x \mapsto n(x)$ is C^1-smooth[3]) and hence F^* is also C^1-smooth.

Observe, that using the notations $D^* \equiv \{(x, h(x)) : x \in R^{N-1}\}, F(\infty) = p$, $h(\infty) \equiv \sup\{\rho \geqq 0 : \operatorname{dist}(F^*(p, -\rho), \partial\Omega) = \rho\}$ and $n(\infty) \equiv$ [the normal vector at p of $\partial\Omega$ directed outward from Ω] we have

$$\Omega = F^*(D \cup D^*) \cup (F(\infty) - (0, h(\infty)] \cdot n(\infty)). \tag{8}$$

For, if y is an arbitrary point of Ω then for some $x \in R^{N-1}$ or $x = \infty$, $F(x)$ is the projection of y on $\partial\Omega$. Thus for some of these x and for some $\xi > 0$ we have dist $(F(x) - \xi \cdot n(x), \partial\Omega) = \operatorname{dist}(F(x) - \xi \cdot n(x), F(x)) = \xi$ and we can write $y = F(x) - \xi \cdot n(x)$. Then the triangle inequality
entails that for all $\rho \in [0, \xi]$. dist $(F(x) - \rho \cdot n(x), \partial\Omega) =$ dist $(F(x) - \rho \cdot n(x), F(x)) = \rho$ holds, which shows (by the definition of h) that $h(x) \geqq \xi$. This proves (8).

Since the set D^* coincides with the graph in $R^{N-1} \times R$ of the function h and since the function h is upper semicontinuous (cf. [6]), we have $\operatorname{vol}_N D^* = 0$. Then from the smoothness of F^* we obtain $\operatorname{vol}_N F^*(D^*) = 0$. Since the mapping F^* is injective on D, for any $f \in L^1(\Omega)$, we have the following integral formula

$$\int_\Omega f = \int_{F^*(D)} f = \int_D f(F^*(x, -\xi)) \operatorname{Jac} F^*(x, -\xi) =$$

$$= \int_{R^{N-1}} \int_0^{h(x)} f(F(x) - \xi \cdot n(x)) \operatorname{Jac} (F(x)) - \xi \cdot n(x) \, d\xi \, dx.$$

To the operations with the matrix $\frac{\partial F^*}{\partial(x, \xi)}$ we fix an arbitrary orthogonal sistem of unit vectors $v^1(x), \ldots, v^{N-1}(x)$ each of which is lying in some principal direction of the surface $\partial\Omega$ at the point $F(x)$ $(x \in R^{N-1})$. Since the rank of the matrix $\frac{\partial F}{\partial x}$ is equal $(N-1)$ everywhere, for any fixed x and for any index i $\in \{1, \ldots, N-1\}$, there exists a unique vector $u^i(x)$ in R^{N-1} such that

$$\frac{\partial F}{\partial x}(x) \cdot u^i(x) = V^i(x) \quad (i = 1, \ldots, N-1;\ x \in R^{N-1}). \tag{11}$$

Now, if $K_i(x)$ denotes the main curvature (with respect to the normal $n(.)$ of $\partial\Omega$ the direction $v^i(x)$ at the point $F(x)$ $(x \in R^{N-1})$, then, as it is well knovn from elementary differential geometry (cf. [5, p. 132]), we have

$$\frac{\partial n}{\partial x}(x) \cdot u^i(x) = K_i(x) \cdot V^i(x) \quad (i = 1, \ldots, N-1;\ x \in R^{N-1}).$$

[3]) Namely, the i-th component of $n(x)$ can be given as

$$\nu \cdot \det\left(\left[\frac{\partial F_k}{\partial x_j}\right]_{j=1,\ i \neq k=1}^{N-1\quad N}\right) \Big/ \sum_{r=1}^{N} \left\{\det\left(\left[\frac{\partial F_k}{\partial x_j}\right]_{j=1,\ r \neq k=1}^{N-1\quad N}\right)\right\}^2,$$

where $\nu = +1$ or -1 independently of x according to the choice of F (cf [5, p. 92]).

Thus, introducing the matrices $U=[u^1, \dots, u^{N-1}]$ (of type $(N-1)\times(N-1)$ and $K=\begin{pmatrix} K_1 & & 0 \\ & \ddots & \\ 0 & & K_{N-1}\end{pmatrix}$ (of type $(N-1)\times(N-1)$), we obtain from (11) and (12)

$$\frac{\partial n}{\partial x} U = \frac{\partial F}{\partial x} UK. \tag{13}$$

Thus we have

$$\left[\frac{\partial F}{\partial x}+\xi\cdot\frac{\partial n}{\partial x}, n\right]=\left[\frac{\partial F}{\partial x}+\xi\cdot\frac{\partial F}{\partial x} UKU^{-1}, n\right]=\left[\frac{\partial F}{\partial x}(I+\xi\, UKU^{-1}), n\right]=$$
$$=\left[\frac{\partial F}{\partial x}, n\right]\cdot\begin{bmatrix} I+\xi UKU^{-1} & 0 \\ 0 & 1\end{bmatrix}.$$

Hence it follows

$$\operatorname{Jac}\left(\frac{\partial F^*}{\partial (x, \xi)}\right)=\left|\det\left[\frac{\partial F}{\partial x}+\xi\cdot\frac{\partial n}{\partial x}, n\right]\right|=\left|\det\left[\frac{\partial F}{\partial x}, n\right]\right|\cdot \tag{14}$$
$$\cdot\left|\det\left[I+\xi\, UKU^{-1}\right]\right|=\left|\det\left[\frac{\partial F}{\partial x}, n\right]\right|\cdot\left|\det\left[I+\xi K\right]\right).$$

Now we derive a formula for the value of $\mathrm{vol}_{N-1}\,\partial\Omega_{-\rho}$ by the aid of (10) and (14). To this we remark, that the function $\rho\mapsto \mathrm{vol}_N\Omega_{-\rho}$ is differentiable from the left and from the right, respectively (at every point $\rho>0$), and the $(N-1)$ dimensional Minkowski content of $\partial\Omega_{-\rho}$ is equal to $-\frac{1}{2}\left(\frac{d^+}{d\rho}+\frac{d^-}{d\rho}\right)\cdot\mathrm{vol}_N\Omega_{-\rho}$ (see [6]). Since any set $\partial\Omega_{-\rho}$ can be given as the Lipschitz continuous image of some compact subset of R^{N-1} [4]), we obtain from a theorem of M. Kneser (see e. g. [7]) that the vol_{N-1} measure of $\partial\Omega_{-\rho}$ coincides with its Minkowski content, i.e.

$$\mathrm{vol}_{N-1}\,\partial\Omega_{-\rho}=-\frac{1}{2}\left(\frac{d^+}{d\rho}+\frac{d^-}{d\rho}\right)\mathrm{vol}_N\,\Omega_{-\rho}. \tag{15}$$

From (10) we obtain

$$\mathrm{vol}_{N-1}\,\Omega_{-\rho}=\int_{R^{N-1}}\int_0^{h(x)} 1_{\Omega_{-\rho}}\cdot(F(x)-\xi\, n(x))\cdot\operatorname{Jac}(F^*(x, -\xi))\,d\xi\,dx. \tag{16}$$

From the definition of the function h we have

$$\Omega_{-\rho}=\{F(x)-\xi\, n(x): \rho<\xi\leqslant h(x),\ \rho\leqslant h(x),\ x\in R^{N-1}\}.$$

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[4]) In fact. if Φ denotes a C^1—mapping from $\overline{B^{N-1}}$ onto ∂B^N, further if $k(y)$ is the outward normal of $\partial\Omega$ at the point $y\in\partial\Omega$. then the mapping $T^*: s\mapsto T(s)-\rho\, k(T(s))$ is a C^1—smooth mapping of ∂B^{N-1} onto some compact subset of R^N which contains $\partial\Omega_{-\rho}$. Therefore. the set $E\equiv(T^*\circ\Phi)^{-1}(\partial\Omega_{-\rho})$ is a compact subset of $\overline{B^{N-1}}$ and we have $\partial\Omega_{-\rho}=(T^*\circ\Phi)(E)$.

Hence

$$1_{\Omega_{-\rho}}(F(x)-\xi n(x))=1_{(\rho,h(x))}(\xi)$$

whenever $0<\xi\leqslant h(x)$. Therefore (16) yields

$$\text{(17)}\qquad \mathrm{vol}_N\Omega_{-\rho}=\int_{R^{N-1}}\int_0^{\max(h(x),\rho)}\mathrm{Jac}(F^*(x,-\xi))\,d\xi\,dx.$$

(17) immediately implies

$$\frac{d^+}{d\rho}\mathrm{vol}_N\Omega_{-\rho}=-\int_{\{x\in R^{N-1}:h(x)>\rho\}}\mathrm{Jac}(F^*(x,-\rho))\,dx=-\int_{R^{N-1}}\mathrm{Jac}(F^*(x,-\rho))\,1_{[o,h(x)]}(\rho)\,dx,$$

(18)

$$\frac{d^-}{d\rho}\mathrm{vol}_N\Omega_{-\rho}=-\int_{\{x\in R^{N-1}:h(x)\geqslant\rho\}}\mathrm{Jac}(F^*(x,-\rho))\,dx=-\int_{R^{N-1}}\mathrm{Jac}(F^*x,\ -\rho))\,1_{[o,h(x)]}(\rho)\,dx.$$

From (14) we have

$$\text{(19)}\qquad \mathrm{Jac}(F^*(x,-\rho))=\left|\det\left[\frac{\partial F}{\partial x},n\right]\right|\cdot\prod_{i=1}^{N-1}(1-\rho\,k_i(x),$$

for any fixed $x\in R^{N-1}$. Recall that for the values of h we have the following estimation

$$\text{(20)}\qquad 0<h(x)\leqslant\frac{1}{k_i(x)}$$

for all indexes i such that $k_i(x)>0$ (see [8, Theorem *B*] or [9]). To complete the proof of Lemma 2, we prove separately the following general statement.

Lemma 3. *Let $P: R\to R$ be such a non-constant polynomial whose roots are all real and $P(0)>0$, $P'(0)\leqslant 0$. Then P has at least one positive root and it decreases in the interval between 0 and its least positive root.*

Having established lemma, we conclude the proof of Lemma 2 as follows. Consider the polynomial $P:\rho\mapsto\prod_{i=1}^{N-1}(1-\rho\,k_i(x))$ for fixed $x\in R^{N-1}$. From our basic assumption on Minkowski curvature of $\partial\Omega$ we have

$$\text{(21)}\qquad P'(0)=-\sum_{i=1}^{N-1}k_i(x)\leqslant 0\,.$$

Therefore we may apply Lemma 3. Hence we see, by taking into consideration (15) (18) (19) and (20), that the function $\rho\mapsto\mathrm{vol}_{N-1}\delta\Omega_{-\rho}$ is monotone decreasing and

$$\text{(22)}\qquad \lim_{\rho\to 0}\mathrm{vol}_{N-1}\partial\Omega_{-\rho}=\int_{R^{N-1}}\mathrm{Jac}(F^*(x,0))\,dx=\mathrm{vol}_{N-1}\partial\Omega.$$

4. Proof of the Lemma 3. It is easy to see using Rolle's theorem, that if all the roots of a polynomial $Q: R \to R$ are real and if $Q(\rho) = a(\rho-\xi_1)^{m_1} \cdots (\rho-\xi_s)^{m_s}$, where $\xi_1 < \cdots < \xi_s$ and $m_1, \ldots, m_s > 0$, then there exist $\eta_1, \ldots, \eta_{s-1}$, b such that $\xi_i < \eta_i < \xi_{i+1}$ $(i=1, \ldots, s-1)$ and $Q'(\rho) = b \cdot (\rho-\xi_1)^{m_1-1} \cdot \ldots \cdot (\rho-\xi_s)^{m_s-1} \cdot (\rho-\eta_1) \cdots (\rho-\eta_{s-1})$.

Therefore the polynomial P has a positive root. In the contrary case our assumptions $P(0) > 0$, $P'(0) \leqslant 0$ imply that P' has *degree* $(P) - 1$ negative roots (with multiplicity). This is impossible because now we necessarily would have $\lim_{\rho \to \infty} P(\rho) = \infty$, whence P' would admit at least *degree* (P) roots (with multiplicity)).

Thus let ξ^* denote the least the least positive root of P, and suppose P is not decreasing on $(0, \xi^*)$. Now for some $\xi \in (0, \xi^*)$ we must have $P'(\xi) > 0$.

But then the relations $P(0) > 0$, $P'(0) \leqslant 0$ $P'(\xi^*) \leqslant 0$ ensure the existence of at least two distinct roots of P' on the interval $[0, \xi^*)$. However, this is impossible since from our beginning remark we can see that the closed interval between any two (not necessarily distinct) roots of P' thus ξ^* cannot be the least among the positive roots of P.

The proof of Lemma 3. and hence also the proof the Theorem is complete.

5. Added in proof. Under the hypothesis of the Theorem, we can prove by the same method the estimate

$$R_j(\Omega) \leqslant \left(j - \frac{1}{2}\right)\pi, \quad (j = 1, 2, \ldots; \ \Omega \in K) \tag{23}$$

where

$$R_j(\Omega) = \Lambda_j(\Omega) \frac{\mathrm{vol}_N \Omega}{\mathrm{vol}_{N-1} \partial \Omega}.$$

It is an open question what are the exact values of $c_j(K) = \sup_{\Omega \in K} R_j(\Omega)$ for $j \neq 1$. Although the estimate (23) does not seem to be exact, it shows, that $\sup_{\Omega \in K} R_j(\Omega) < \infty (j = 1, 2, \ldots)$, which is not trivial apriori.

It is worth to remark, that the order behaviour from above of the quantities $\{\Lambda_j(\Omega)\}_{j=1}^{\infty}$ for an arbitarily fixed domain Ω is exactly clarified by the following famous theorem of V. A. Il'in [12]: Given any domain Ω in R^N there exist a constant $C(\Omega)$ depending only on Ω such that for each orthonormal system $\{u_j\} \subset L^2(\Omega)$ formed by the eigenfunctions of the Laplacian with respective eigenvalues $\{\Lambda_j\}_1^{\infty}$, the estimate

$$0 < \Lambda_j \leqslant C(\Omega) \cdot j^{\frac{2}{N}} (j = 1, 2, \ldots)$$

holds. This estimate cannot be improved.

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