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SHAPE FIBRATIONS FOR COMPACT HAUSDORFF SPACES

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1. Introduction.

In [10] S. Mardešić has introduced the notion of a shape fibration for maps between arbitrary topological spaces. The author in [4] gives two alternative definitions of shape fibrations and proves that they are equivalent to the original definition. Some further properties of shape fibrations in the non-compact case are established in [5].

In this paper we show that the definition of shape fibration between compact Hausdorff spaces can be simplified in the sense that the approximate homotopy lifting property (AHLP) is replaced by the homotopy lifting property (HLP). The main results of this paper are Theorems 4.3 and 5.1. The first one asserts that the pull-back of a shape fibration between compact Hausdorff spaces is again a shape fibration. This generalizes the analogous result for compact metric spaces due to M. Jani [7] and A. Matsumoto [15]. The second one asserts that whenever $p: E\to B$ is a shape fibration of compact Hausdorff spaces and $x, y\in B$ are points connected by an arc in B, then $Sh(X)=Sh(Y)$ where $X=p^{-1}(x)$, $Y=p^{-1}(y)$. This generalizes the analogous results for compact metric spaces due to S. Mardešić and T. B. Rushing [11].

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2. Preliminaries.

In this section we give the definitions of the basic notions and we state some known facts about them needed in the sequel

2.1. *A map of inverse systems* $\mathbf{p}=(p_\mu, \pi): \mathbf{E}=(E_\lambda, q_{\lambda\lambda'}, \Lambda)\to \mathbf{B}=(B_\mu, r_{\mu\mu'}, M)$ in the category Top consists of a function $\pi: M\to\Lambda$ between directed sets and of a family of maps $p_\mu: E_{\pi(\mu)}\to B_\mu$, $\mu\in M$, such that for $\mu\leqslant\mu'$ there exists a $\lambda\in\Lambda$, $\lambda\geqslant\pi(\mu), \pi(\mu')$ satisfying

$$(1)\qquad p_\mu\, q_{\pi(\mu)\lambda}=r_{\mu\mu'}\, p_{\mu'}\, q_{\pi(\mu')\lambda}.$$

2.2. If Λ is a singleton, then a map of systems $\mathbf{p}=(p_\mu, M): E\to\mathbf{B}$ consists of a family of maps $p_\mu: E\to B_\mu$, $\mu\in M$, such that $p_\mu=r_{\mu\mu'}\, p_{\mu'}$, $\mu\leqslant\mu'$.

2.3. If $M=\Lambda$ and $\pi=1_\Lambda$, then instead of (1) we can assume

$$p_\lambda q_{\lambda\lambda'}=r_{\lambda\lambda'}p_{\lambda'},\quad \lambda\leqslant\lambda';$$

in this case we say that $\mathbf{p}=(p_\lambda, 1_\Lambda):\mathbf{E}\to\mathbf{B}$ is a *level map of systems.*

2.4. The *composition* of maps of systems $\mathbf{p}=(p_\mu, \pi):\mathbf{E}\to\mathbf{B}$ and $\mathbf{p}'=(p'_\nu, \pi'):\mathbf{B}\to\mathbf{C}=(C_\nu, s_{\nu\nu'}, N)$ is the map of systems $\mathbf{h}=(h_\nu, h):\mathbf{E}\to\mathbf{C}$, where $h=\pi\pi':N\to\Lambda$ and $h_\nu=p'_\nu p_{\pi'(\nu)}:E_{h(\nu)}\to C_\nu$, $\nu\in N$.

2.5. Two maps of systems $\mathbf{p}=(p_\mu, \pi)$, $\mathbf{p}'=(p'_\mu, \pi'):\mathbf{E}\to\mathbf{B}$ are said to be *equivalent*, $\mathbf{p}\sim\mathbf{p}'$, if for each $\mu\in M$ there is a $\lambda\in\Lambda$, $\lambda\geqslant\pi(\mu)$, $\pi'(\mu)$ such that

$$p_\mu q_{\pi(\mu)\lambda}=p'_\mu q_{\pi'(\mu)\lambda}.$$

We will denote by Inv-Top the category whose objects are inverse systems in Top and whose morphisms are maps of systems. We will denote by pro-Top the category whose objects are inverse systems in Top and whose morphisms are equivalence classes [**p**] of maps of systems under the relation $\sim$.

Definition 2.6. [10]. A *resolution of a space E* is a map of systems $\mathbf{q}=(q_\lambda, \Lambda):E\to\mathbf{E}=(E_\lambda, q_{\lambda\lambda'}, \Lambda)$ satisfying the following two conditions:

(R1) Let P be a polyhedron, $\mathcal{V}$ an open covering of P and $f:E\to P$ a map. Then there is a $\lambda\in\Lambda$ and a map $f_\lambda:E_\lambda\to P$ such that the maps $f_\lambda q_\lambda$ and f are $\mathcal{V}$-near, which we denote by $(f_\lambda q_\lambda, f)\leqslant\mathcal{V}$.

(R2) Let P be a dolyhedron and let $\mathcal{V}$ be an open covering of P. Then there is an open covering $\mathcal{V}'$ of P with the following property: If $\lambda\in\Lambda$ and $f, f':E_\lambda\to P$ are maps satisfying $(fq_\lambda, f'q_\lambda)\leqslant\mathcal{V}'$, then there is a $\lambda'\geqslant\lambda$ such that $(fq_{\lambda\lambda'}, f'q_{\lambda\lambda'})\leqslant\mathcal{V}$.

If for a resolution $\mathbf{q}:E\to\mathbf{E}$ all E_λ are polyhedra (*ANR*-spaces), we call **q** a *polyhedral* (*ANR*)*-resolution.*

Definition 2.7. [4] A resolution of a map $p:E\to B$ is a triple (**q**, **r**, **p**) which consists of resolutions $\mathbf{q}=(q_\lambda, \Lambda):E\to\mathbf{E}$ and $\mathbf{r}=(r_\mu, M):\mathbf{B}\to\mathbf{B}=(B_\mu, r_{\mu\mu'}, M)$ of spaces E and B respectively and of a map of systems $\mathbf{p}=(p_\mu, \pi):\mathbf{E}\to\mathbf{B}$ such that $\mathbf{pq}=\mathbf{r}p$, i.e. $p_\mu q_{\pi(\mu)}=r_\mu p$ for each $\mu\in M$.

If **q** and **r** are polyhedral (*ANR*)-resolutions of E and B, then (**q**, **r**, **p**) is called a polyhedral (*ANR*) resolution of p. If $\mathbf{p}=(p_\lambda, 1_\Lambda):\mathbf{E}\to\mathbf{B}=(B_\lambda, r_{\lambda\lambda'}, \Lambda)$ is a level map of systems, then (**q**, **r**, **p**) is called a *level resolution* of p. In this case we have $p_\lambda q_\lambda=r_\lambda p$ for each $\lambda\in\Lambda$.

In [10] the following two conditions were shown to be sufficient in order that $\mathbf{q}:E\to\mathbf{E}$ be a resolution:

(B1) For any normal covering $\mathcal{U}$ of E there is a $\lambda\in\Lambda$ and a normal covering $\mathcal{U}_\lambda$ of E_λ such that $q_\lambda^{-1}(\mathcal{U}_\lambda)$ refines $\mathcal{U}$, which we denote by $q_\lambda^{-1}(\mathcal{U}_\lambda)\geqslant\mathcal{U}$.

(B2) For each $\lambda\in\Lambda$ and for each open neighborhood U of cl $(q_\lambda(E))$ in E_λ there is a $\lambda'\geqslant\lambda$ such that $q_{\lambda\lambda'}(E_{\lambda'})\subseteq U$.

In [10] it was shown also that for normal E'_λ s conditions (B1) and (B2) are necessary in order that $\mathbf{q}:E\to\mathbf{E}$ be a resolution. Consequently, every polyhedral resolution of a space E has properties (B1) and (B2).

Theorem 2.8. *If E is a compact Hausdorff space and* $\mathbf{E}$ *is an inverse system of compact Hausdorff spaces, then* $\mathbf{q}: E \to \mathbf{E}$ *is a resolution of E if and only if* $\mathbf{q}: E \to \mathbf{E}$ *is an inverse limit, i.e. lim* $\mathbf{E} = (E, \mathbf{q})$. ([14], § 6, *Theorem* 1 *or* [10] *Theorems* 7 *and* 8).

Definition 2.9. [4]. A level map of systems $\mathbf{p} = (p_\lambda, 1_\Lambda): \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda) \to \mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$ is said to have the *homotopy lifting property* (*HLP*) with respect to a class $\mathcal{X}$ if for each $X \in \mathcal{X}$ and for each $\lambda \in \Lambda$ there is a $\lambda' \geqslant \lambda$ with the following property: whenever $h: X \to E_{\lambda'}$ and $H: X \times I \to B_{\lambda'}$ are maps satisfying $p_{\lambda'} h = H_0$, then there is a homotopy $\tilde{H}: X \times I \to E_\lambda$ such that $q_{\lambda\lambda'} h = \tilde{H}_0$ and $p_\lambda \tilde{H} = r_{\lambda\lambda'} H$. λ' is called a lifting index for λ.

Definition 2.10. [4]. A level map $p: \mathbf{E} \to \mathbf{B}$ has the *approximate homotopy lifting property* (*AHLP*) with respect to a class $\mathcal{X}$ if for each $\lambda \in \Lambda$ and for any two normal coverings $\mathcal{U}$, $\mathcal{V}$ of E_λ and B_λ respectively, there is a $\lambda' \geqslant \lambda$ and a normal covering $\mathcal{V}'$ of $B_{\lambda'}$ with the following property: whenever $X \in \mathcal{X}$ and $h: X \to E_{\lambda'}$, $H: X \times I \to B_{\lambda'}$ are maps satisfying $(p_{\lambda'} h, H_0) \leqslant \mathcal{V}'$ then there is a homotopy $\tilde{H}: X \times I \to E_\lambda$ such that

$$(q_{\lambda\lambda'} h, \tilde{H}_0) \leqslant \mathcal{U} \text{ and } (p_\lambda \tilde{H}, r_{\lambda\lambda'} H) \leqslant \mathcal{V}.$$

λ' and $\mathcal{V}'$ are called lifting index and lifting mesh for λ, $\mathcal{U}$, $\mathcal{V}$, respectively.

Definition 2.11. [4]. A map of topological spaces $p: E \to B$ is called a *shape fibration* provided there is a polyhedral level resolution $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ of p such that the level map of systems $\mathbf{p}: \mathbf{E} \to \mathbf{B}$ has the *AHLP* with respect to the class of all topological spaces.

3. On resolutions of spaces and maps

In this section we establish some facts about resolutions of spaces and maps, which are needed in the next sections of this paper.

Similarly as in [4], Theorem 2.3, we can prove the following theorem.

Theorem 3.1. *Let $p: E \to B$ and $f: C \to B$ be maps of topological spaces. Then there are polyhedral resolutions* $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ *and* $(\mathbf{s}, \mathbf{r}, \mathbf{f})$ *of p and f respectively.*

Proof. Let Γ be the set of all normal coverings γ of B. For each $\gamma \in \Gamma$ we choose a locally finite partition of unity $(\Psi_V, V \in \gamma)$ subordinated to γ. Let $N(\gamma)$ denote the nerv of γ and let $B'_\gamma = |N(\gamma)|$ be the carrier of $N(\gamma)$. Let $r'_\gamma: B \to B'_\gamma$ be the canonical map determined by $(\Psi_V, V \in \gamma)$ ([3], VIII. Theorem 5.4). It maps $y \in B$ to the point $r'_\gamma(y)$ whose barycentric coordinate with respect to the vertex V equals $\Psi_V(y)$.

For each $\gamma \in \Gamma$, $p^{-1}(\gamma) = \{p^{-1}(V): V \in \gamma\}$ and $f^{-1}(\gamma) = \{f^{-1}(V): V \in \gamma\}$ are normal coverings of E and C respectively. Let $E'_\gamma = |N(p^{-1}(\gamma))|$, $C'_\gamma = |N(f^{-1}(\gamma))|$, $\varphi_V = \Psi_V p$ and $\phi_V = \Psi_V f$. Clearly, $(\varphi_V, V \in \gamma)$ and $(\phi_V, V \in \gamma)$

are locally finite partitions of unity subordinated to the coverings $p^{-1}(\gamma)$ of E and $f^{-1}(\gamma)$ of C, respectively. These partitions determine, as above, canonical maps $q'_\gamma : E \to E'_\gamma$ and $s'_\gamma : C \to C'_\gamma$ respectively. We define a simplicial map $p'_\gamma : E'_\gamma \to B'_\gamma$ by sending a vertex $V \in \gamma$ of $N(p^{-1}(\gamma))$, $p^{-1}(V) \neq \varnothing$, to the vertex V of $N(\gamma)$. The simplicial map $f'_\gamma : C'_\gamma \to B'_\gamma$ is defined similarly. One readily sees that

$$p'_\gamma q'_\gamma = r'_\gamma p \text{ and } f'_\gamma s'_\gamma = r'_\gamma f.$$

Let A denote the set of all normal coverings of E not of the form $p^{-1}(\gamma)$, $\gamma \in \Gamma$, and let P' be the set of all normal coverings of C not of the form $f^{-1}(\gamma)$, $\gamma \in \Gamma$. For each $\alpha \in A'$ and eacq $\beta \in P'$ we choose a locally finite partitions of unity $(\varphi_U, U \in \alpha)$ and $(\Phi_W, W \in \beta)$ subordinated to α and β respectively. We now put $A = A' \cup \Gamma$ and $P = P' \cup \Gamma$. A and P are the sets of all normal coverings of E and C, respectively. We define $\pi' : \Gamma \to A$ and $\varphi' : \Gamma \to P$ to be inclusion maps.

Now let I denote the set of all finite subsets of Γ ordered by inclusion. Clearly, I is a directed and cofinite set. For $i = \{\gamma_1, \gamma_2, \cdots \gamma_n\}$ we put $B'_i = |N(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n)|$, where $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n = \{V_1 \cap V_2 \cap \cdots \cap V_n : (V_1, V, \ldots, V_n) \in \gamma_1 \times \gamma_2 \cdots \gamma_n\}$ is a normal covering of B. If $i \leqslant i' = \{\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots, \gamma_m\}$, by $r'_{ii'} : B'_{i'} \to B'_i$ we denote the simplicial map which sends the vertex $(V_1 V_2, \ldots, V_n, \ldots, V_m)$ of $N(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n \wedge \cdots \wedge \gamma_m)$, $\bigcap_{j=1}^{m} V_j \neq \varnothing$, to the vertex $(V_1, V_2, \ldots, V_n)$ of $N(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n)$. For $i = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ let $r'_i : B \to B'_i$ be the canonical map tetermined by the partition of unity $(\Psi_{(V_1 V_2, \ldots, V_n)}, (V_1, V_2, \ldots, V_n) \in \gamma_1 \times \gamma_2 \cdots \gamma_n)$ subordinated to the cover $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n$, where $\Psi_{(V_1, V_2, \ldots, V_n)} = \Psi_{V_1} \cdot \Psi_{V_2} \cdots \Psi_{V_n}$. Then one has

$$r'_{ii'} r'_{i'i''} = r'_{ii''}, \quad i \leqslant i' \leqslant i''$$

$$r'_{ii'} r'_{i'} = r'_i, \quad i \leqslant i'.$$

Let J and K denote the sets of all finite subsets of A and P, respectively, ordered by inclusion. For each $j = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in J$ and for each $k = \{\beta_1, \beta_2, \ldots, \beta_n\} \in K$ we put $E'_j = |N(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n)|$ and $C'_k = |N(\beta_1 \wedge \wedge \beta_2 \wedge \cdots \wedge \beta_n)|$. The maps $q'_{jj'} : E'_{j'} \to E'_j$, $j \leqslant j'$, $q'_j : E \to E'_j$ $j \in J$, $s'_{kk'} : C'_{k'} \to C'_k$, $k \leqslant k'$, and $s'_k : C \to C'_k$, $k \in K$, are defined similarly as the maps $r'_{ii'}$ and r'_i. Then one has

$$q'_{jj'} q'_{j'j''} = q'_{jj''}, \ j \leqslant j' \leqslant j'',$$

$$s'_{kk'} s'_{k'k''} = s'_{kk''}, \ k \leqslant k' \leqslant k'',$$

$$q'_{jj'} q'_{j'} = q'_j, \ j \leqslant j',$$

$$s'_{kk'} s'_{k'} = s'_k, \ k \leqslant k'.$$

The inclusions π' and φ' are extented to increasing functions $\pi': I \to J$ and $\varphi': I \to K$, respectively, by puting $\pi'(i) = i$ and $\varphi'(i) = i$ for each $i = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \in I$. Also, for each $i = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \in I$ we define a simplicial map $p'_i: E'_{\pi'(i)} = E'_i \to B'_i$ by assigning to each vertex $(V_1, V_2, \ldots, V_n)$ of $N(p^{-1}(\gamma_1) \wedge \cdots \wedge p^{-1}(\gamma_n))$, $p^{-1}(V_1 \cap V_2 \cap \cdots \cap V_n) \neq \varnothing$, the vertex $(V_1, V_2, \ldots, V_n)$ of $N(\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n)$. Similarly, one defines the simplicial map $f'_i: C'_{\varphi'(i)} = C'_i \to B'_i$ for each $i \in I$ Then one has

$$p'_i q'_{\pi'(i)\pi'(i')} = r'_{ii'} p'_{i'}, \quad i \leqslant i';$$

$$f'_i s'_{\varphi'(i)\varphi'(i')} = r'_{ii'} f'_{i'}, \quad i \leqslant i';$$

$$p'_i q'_{\pi'(i)} = r'_i p, \quad i \in I;$$

$$f'_i s'_{\varphi'(i)} = r'_i f, \quad i \in I.$$

We have thus defined inverse systems of polyhedra $\mathbf{E}' = (E'_j, q'_{jj'}, J)$, $\mathbf{B}' = (B'_i\ r'_{ii'}, I)$, $\mathbf{C}' = (C'_k, s'_{kk'}, K)$ and maps of systems $\mathbf{q}' = (q'_j, J): E \to \mathbf{E}'$, $\mathbf{r}' = (r'_i, I): B \to \mathbf{B}'$, $\mathbf{s}' = (s'_k, K): C \to \mathbf{C}'$, $\mathbf{p}' = (p'_i, \pi'): \mathbf{E}' \to \mathbf{B}'$, $\mathbf{f}' = (f'_i, \varphi'): \mathbf{C} \to \mathbf{B}$ for which one has

$$\mathbf{p}'\,\mathbf{q}' = \mathbf{r}'\,p \text{ and } \mathbf{f}'\ \mathbf{s}' = \mathbf{r}' f.$$

As in the proof of [4], Theorem 2.3. it is shown that $\mathbf{q}'$, $\mathbf{r}'$ and $\mathbf{s}'$ satisfy condition (B1). In order to odtain also condition (B2) the systems $\mathbf{E}'$, $\mathbf{B}'$, $\mathbf{C}'$ are replaced, as in the proof of [4], Theorem 2.3, by some larger polyhedral systems $\mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$, $\mathbf{B} = (B_\mu, r_{\mu\mu'}, M)$, $\mathbf{C} = (C_\nu, s_{\nu\nu'}, N)$. These systems contain beside the members E'_j of $\mathbf{E}'$, B'_i of $\mathbf{B}'$ and C'_k of $\mathbf{C}'$, also closed polyhedral neighborhoods of $\mathrm{cl}(q'_j(E))$ in E'_j, of $cl(r'_i(B))$ in B'_i and of $cl(s'_k(C))$ in C'_k, respectively. Furthermore maps of systems $\mathbf{q} = (q_\lambda, \Lambda): E \to \mathbf{E}$, $\mathbf{r} = (r_\mu, M): B \to \mathbf{B}$, $\mathbf{s} = (s_\nu, N): C \to \mathbf{C}$, $\mathbf{p} = (p_\mu, \pi): \mathbf{E} \to \mathbf{B}$ and $\mathbf{f} = (f_\mu, \varphi): \mathbf{C} \to \mathbf{B}$ for which $\mathbf{p}\,\mathbf{q} = \mathbf{r}\,p$ and $\mathbf{f}\,\mathbf{s} = \mathbf{r} f$ hold are obtained as in the proof of [4], Theorem 2.3. Now $\mathbf{q}$, $\mathbf{r}$, $\mathbf{s}$ satisfy both conditions (B1) and (B2). Consequently, $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ and $(\mathbf{s}, \mathbf{r}, \mathbf{f})$ are polyhedral resolutions of p and f respectively.

It is well known that every open covering of a compact Hausdorff space is a normal covering of that space. Also, every open covering of such a space admits a finite subcovering which refines it. In view of this and by the proof of Theorem 3.1 we obtain the following theorem.

Theorem 3.2. *If $p: E \to B$ and $f: C \to B$ are maps between compact Hausdorff spaces, then there are compact polyhedral resolutions* $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ *and* $(\mathbf{s}, \mathbf{r}, \mathbf{f})$ *of p and f, respectively.*

Indeed, if in the proof of Theorem 3.1 we take for Γ the set of all finite open coverings of B, then $B'_\gamma = |N(\gamma)|$, $E'_\gamma = |N(p^{-1}(\gamma))|$, $C'_\gamma = |N(f^{-1}(\gamma))|$ are compact polyhedra for each $\gamma \in \Gamma$. Also, in that proof for A' and P' we take the sets of all finite open coverings of E and C respectively, not of the form $p^{-1}(\gamma)$ and $f^{-1}(\gamma)$, $\gamma \in \Gamma$. Then $A = A' \cup \Gamma$ and $P = P' \cup \Gamma$ are the sets of all finite open coverings of E and C respectively. Thus, for $j=\{\alpha_1, \alpha_2, \ldots, \alpha_n\}\in J$, $i=\{\gamma_1, \gamma_2, \ldots, \gamma_n\}\in I$ and $k=\{\beta_1, \beta_2, \ldots, \beta_n\}\in K$, $E'_j=|N(\alpha_1\wedge\cdots\wedge\alpha_n)|$, $B'_i=|N(\gamma_1\wedge\cdots\wedge\gamma_n)|$ and $C'_k=N(\beta_1\wedge\cdots\wedge\beta_n)$ are compact polyhedra. Now E_λ, B_μ, C_ν are compact polyhedia because they are closed polyhedral neighborhoods in E'_j, B'_i, and B'_k respectively. Consequently. $\mathbf{E}$, $\mathbf{B}$, $\mathbf{C}$ are compact polyhedral systems and $(\mathbf{q}, \mathbf{r}, \mathbf{p})$, $(\mathbf{s}, \mathbf{r}, \mathbf{f})$ are compact polyhedral resolutions of p and f respectively.

Similarly as in [4], Theorem 4.8 and [4], Lemma 4.9, we can prove the next theorem and lemma.

Theorem 3.3. *Let* $\mathbf{p}=(p_\mu, \pi):\mathbf{E}=(E_\lambda, q_{\lambda\lambda'}, \Lambda)\to\mathbf{B}=(B_\mu, r_{\mu\mu'}, M)$ *and* $\mathbf{f}=(f_\mu, \varphi):\mathbf{C}=(C_\nu; s_{\nu\nu'}, N)\to\mathbf{B}$ *be maps of systems. Then there are inverse systems* $\mathbf{E}'=(E'_\alpha, q'_{\alpha\alpha'}, A)$, $\mathbf{B}'=(B'_\alpha, r'_{\alpha\alpha'}, A)$ *and* $\mathbf{C}'=(C'_\alpha, s'_{\alpha\alpha'}, A)$ *over the same cofinite index set* A, *there are level maps of systems* $\mathbf{p}'=(p'_\alpha, 1_A):\mathbf{E}'\to\mathbf{B}'$ *and* $\mathbf{f}'=(f'_\alpha, 1_A):\mathbf{C}'\to\mathbf{B}'$ *and there are isomorphisms* $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *in pro-Top such that the following diagram commutes in pro-Top*

$$\begin{array}{ccccc} \mathbf{E} & \xrightarrow{\mathbf{p}} & \mathbf{B} & \xleftarrow{\mathbf{p}} & \mathbf{C} \\ \mathbf{i}\downarrow & \sim & \mathbf{j}\downarrow & \sim & \downarrow \\ \mathbf{E}' & \xrightarrow[\mathbf{p}']{} & \mathbf{B} & \xleftarrow[\mathbf{p}']{} & \mathbf{C}' \end{array}$$

Lemma 3.4. *Let* $p:E\to B$ *and* $f:C\to B$ *be maps of topological spaces. If* $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ *and* $(\mathbf{s}, \mathbf{r}, \mathbf{f})$ *are resolutions of* p *and* f *respectively, and* $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{p}', \mathbf{f}'$ *are as in Theorem* 3.3, *then* $(\mathbf{q}', \mathbf{r}', \mathbf{p}')=(\mathbf{i}\,\mathbf{q}, \mathbf{j}\,\mathbf{r}, \mathbf{p}')$ *and* $(\mathbf{s}', \mathbf{r}', \mathbf{t}')=(\mathbf{k}\,\mathbf{s}, \mathbf{j}\,\mathbf{r}, \mathbf{f}')$ *are level resolutions of* p *and* f *respectively.*

From Theorem 3.2 and Lemma 3.4 we obtain the follwing corollary:

Corollary 3.5. *If* $p:E\to B$ *and* $f:C\to B$ *are maps between compact Hausdorff spaces, then there are compact polyhedral level resolutions* $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ *and* $(\mathbf{s}, \mathbf{r}, \mathbf{f})$ *of* p *and* f, *respectively.*

The following fact is needed in the sequel.

Theorem 3.6. *Let* $\mathbf{r}=(r_\lambda, \Lambda):B\to\mathbf{B}=(B_\lambda, r_{\lambda\lambda'}, \Lambda)$ *be a locally compact polyhedral resolution of a topological space* B *with* Λ *cofinite and let* B_0 *be a compact and P-embedded subset of* B. *If* $\mathbf{r}_0=(r_\lambda|B_0, \Lambda):B_0\to\mathbf{B}_0=(B_{0\lambda}, r_{\lambda\lambda'}|B_{0\lambda'}, \Lambda)$ *is a resolution of* B_0 *such that each* $B_{0\lambda}$ *is a compact subset of* B_λ *with* $r_\lambda(B_0)\subseteq B_{0\lambda}$, *then for each* $\lambda\in\Lambda$ *there is a closed polyhedral neighborhood* B'_λ *of* $B_{0\lambda}$ *in* B_λ *such that*

$$r_{\lambda\lambda'}(B'_{\lambda'})\subseteq \mathrm{Int}\, B'_\lambda,\ \lambda<\lambda', \tag{1}$$

and such that $\mathbf{r}'=(r_\lambda|B_0, \Lambda):B_0\to\mathbf{B}'=(B'_\lambda, r_{\lambda\lambda'}|B'_{\lambda'}, \Lambda)$ *is a locally compact polyhedral resolution of* B_0.

Recall that a subset B_0 is *P-embedded* in B if for every normal covering $\mathcal{U}_0$ of B_0 there is a normal covering $\mathcal{U}$ of B such that $\mathcal{U}|A=\{U\cap A: U\in\mathcal{U}\}$ refines $\mathcal{U}_0$ ([1], Theorem 14.7, p. 178).

Proof of Theorem 3.6. For each $\lambda \in \Lambda$, B_λ is a locally compact polyhedron and therefore a metric space (in fact an *ANR* space) ([2], *p*. 80, 81). Thus, for each $\lambda \in \Lambda$, $B_{0\lambda}$ is a compact subset od a metric space B_λ and consequently it admits a decreasing secuence of open neighborhoods

$$U_{\lambda,1} \supseteq U_{\lambda,2} \supseteq \cdots \supseteq U_{\lambda,n} \supseteq \cdots$$

in B_λ such that for every open neighborhood U of $B_{0\lambda}$ in B_λ there is a number n with $U_{\lambda,n} \subseteq U$.

Using induction on the number of predecessors of λ different from λ, one can assign to each $\lambda \in \Lambda$ a closed polyhedral neighborhood B'_λ of $B_{0\lambda}$ in B_λ. Indeed, let Λ_k be the set of all $\lambda \in \Lambda$ with exactly k predecessors different from λ. If $\lambda \in \Lambda_0$, we take for B'_λ an arbitrary closed polyhedral neighborhood of $B_{0\lambda}$ in B_λ such that $B'_\lambda \subseteq U_{\lambda,1}$ ([14], § 6, Lemma 7). Now assume that we have already defined B'_λ satisfying (1) for all $\lambda, \lambda' \in \bigcup_{j=0}^{k-1} \Lambda_j$ and such that

$$B'_{\lambda'} \subseteq r^{-1}_{\lambda\lambda'} (\operatorname{Int} B'_\lambda \cap U_{\lambda,k}), \ \lambda < \lambda', \ \lambda' \in \bigcup_{j=0}^{k-1} \Lambda_j .$$

Let $\lambda \in \Lambda_k$ and let $\lambda_1, \lambda_2, \ldots, \lambda_k < \lambda$ be all the predecessors of λ different from λ. Then $\lambda_i \in \bigcup_{j=0}^{k-1} \Lambda_j$ for each $i = 1, 2, \ldots, k$, and the closed polyhedral neighborhoods B'_{λ_i} have already been constructed. Notice that for each $i = 1, 2, \ldots, k$, $r^{-1}_{\lambda_i\lambda} (\operatorname{Int} B'_{\lambda_i} \cap U_{\lambda_i, k+1})$ is an open neighborhood of $B_{0\lambda}$ in B_λ. Hence, the same is true of $\bigcap_{i=1}^{k} r^{-1}_{\lambda_i\lambda} (\operatorname{Int} B'_{\lambda_i} \cap U_{\lambda_i,k+1})$. Therefore, there is a closed polyhedral neighborhood B'_λ of $B_{0\lambda}$ in B_λ such that

$$B'_\lambda \subseteq \bigcap_{i=1}^{k} r^{-1}_{\lambda_i\lambda} (\operatorname{Int} B'_{\lambda_i} \cap U_{\lambda_i, k+1}). \tag{2}$$

From (2) it is clear that (1) holds for $\lambda' \in \Lambda_k$, and so the inductive construction of the neighborhoods B'_λ is completed.

We now prove that $\mathbf{r}' : B_0 \to \mathbf{B}' = (B'_\lambda, r_{\lambda\lambda'} \mid B'_{\lambda'}, \Lambda)$ is a resolution of B_0. To do this it is sufficient to verify that conditions (B1) and (B2) are fulfilled.

(B1). Let $\mathcal{U}_0$ be a normal covering of B_0. Since B_0 is *P*-embedded in B, there is a normal covering $\mathcal{U}$ of B such that $\mathcal{U} \mid B_0$ refines $\mathcal{U}_0$. $\mathbf{r} : B \to \mathbf{B}$ being a polyhedral resolutions of B has property (B1), and thus, there is a $\lambda \in \Lambda$ and an open covering $\mathcal{U}_\lambda$ of B_λ such that $r_\lambda^{-1} (\mathcal{U}_\lambda)$ refines $\mathcal{U}$. Then $\mathcal{U}_{0\lambda} = \mathcal{U}_\lambda \mid B'_\lambda$ is an open covering of B'_λ and $(r_\lambda \mid B_0)^{-1} (\mathcal{U}_{0\lambda})$ refines $\mathcal{U}_0$.

(B2). Let V be an open neighborhood of $cl(r_\lambda(B_0))$ in B'_λ. Then $V \cap B_{0\lambda}$ is an open neighborhood of $cl(r_\lambda(B_0))$ in $B_{0\lambda}$. Since each $B_{0\lambda}$ is a normal space, the resolution $\mathbf{r}_0 : B \to \mathbf{B}_0$ has property (B2). Consequently, there is a

$\lambda' \geqslant \lambda$ such that $r_{\lambda\lambda'}(B_{0\lambda'}) \subseteq V \cap B_{0\lambda} \subseteq V$, i.e. $B_{0\lambda'} \subseteq r^{-1}_{\lambda\lambda'}(V)$. Then $r^{-1}_{\lambda\lambda'}(V) \cap \operatorname{Int} B'_{\lambda'}$ is an open open neighqorhood of $B_{0\lambda'}$ in $B_{\lambda'}$. Hence, by the choice of the decreasing sequence of neighborhoods $\{U_{\lambda',n}\}$, there is a number n such that

$$U_{\lambda',n} \subseteq r^{-1}_{\lambda\lambda'}(V) \cap \operatorname{Int} B'_{\lambda'}.$$

Let now $\lambda'' \geqslant \lambda'$ be an index with at least $(n-1)$ predecessors different from λ''. Then, by (2), $B'_{\lambda''} \subseteq r^{-1}_{\lambda'\lambda''}(\operatorname{Int} B'_{\lambda'} \cap U_{\lambda',n})$, which implies that

$$r_{\lambda'\lambda''}(B'_{\lambda''}) \subseteq U_{\lambda',n} \subseteq r^{-1}_{\lambda\lambda'}(V) \cap \operatorname{Int} B'_{\lambda'} \subseteq r^{-1}_{\lambda\lambda'}(V)$$

i.e. $r_{\lambda\lambda''}(B'_{\lambda''}) \subseteq V$.

Since every closed subset B_0 of a compact Hausdorff space B is P-embedded in B and since (2.8) holds, we immediately obtain, by Theorem 3.6, the following fact:

Theorem 3.7. *Let B be a compact Hausdorff space and let lim $\mathbf{B} = (B, \mathbf{r})$, where $\mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$ is an inverse system of compact polyhedra with Λ cofinite. If B_0 is a closed set of B and lim $\mathbf{B}_0 = (B_0, \mathbf{r}_0)$ where each $B_{0\lambda}$ is a closed subset of B_λ with cl $(r_\lambda(B)) \subseteq B_{0\lambda}$ and $r_{0\lambda} = r_\lambda | B_0$, then for each $\lambda \in \Lambda$ there is a closed polyhedral neighborhood B'_λ of $B_{0\lambda}$ in B_λ such that*

$$r_{\lambda\lambda'}(B'_{\lambda'}) \subseteq \operatorname{Int} B'_\lambda, \ \lambda < \lambda',$$

and such that lim $\mathbf{B}' = (B_0, \mathbf{r}')$, where $\mathbf{r}' = (r_\lambda | B_0, \Lambda): B_0 \to \mathbf{B}' = (B'_\lambda, r_{\lambda\lambda'} | B'_{\lambda'}, \Lambda)$.

Proposition 3.8. *If $\mathbf{q} = (q_\lambda, \Lambda): E \to \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$ and $\mathbf{r} = (r_\lambda, \Lambda)$: $: B \to \mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$ are resolutions of compact Hausdorff spaces E and B respectively, then*

$$\mathbf{q} \times \mathbf{r} = (q_\lambda \times r_\lambda, \Lambda): E \times B \to \mathbf{E} \times \mathbf{B} = (E_\lambda \times B_\lambda, q_{\lambda\lambda'} \times r_{\lambda\lambda'}, \Lambda)$$

is a resolution of $\mathbf{E} \times \mathbf{B}$.

Proof. It is sufficient to verify conditions (B1) and (B2) for $\mathbf{q} \times \mathbf{r}$.

(B1). Let $\mathcal{W}$ be an open covering of $E \times B$. Then there are open coverings $\mathcal{U}$ and $\mathcal{V}$ of E and B respectively, such that $\mathcal{U} \times \mathcal{V} = \{U \times V : U \in \mathcal{U}; V \in \mathcal{V}\}$ refines $\mathcal{W}$ (see the proof of [8], Theorem 8, p. 233). Since $\mathbf{q}$ and $\mathbf{r}$ have property (B1), there are $\lambda', \lambda'' \in \Lambda$ and open coverings $\mathcal{U}_{\lambda'}, \mathcal{V}_{\lambda''}$ of $E_{\lambda'}, B_{\lambda''}$ respectively, such that $q^{-1}_{\lambda'}(\mathcal{U}_{\lambda'})$ refines $\mathcal{U}$ and $r^{-1}_{\lambda''}(\mathcal{V}_{\lambda''})$ refines $\mathcal{V}$. Let $\lambda \geqslant \lambda'$, λ''. Then $\mathcal{W}_\lambda = q^{-1}_{\lambda'\lambda}(\mathcal{U}_{\lambda'}) \times r^{-1}_{\lambda''\lambda}(\mathcal{V}_{\lambda''})$ is an open covering of $E_\lambda \times B_\lambda$ which refines $\mathcal{W}$.

(B2). Let W be an open neighborhood of $cl\left((q_\lambda \times r_\lambda)(E \times B)\right) = cl\left(q_\lambda(E)\right) \times \times cl\left(r_\lambda(B)\right)$ in $E_\lambda \times B_\lambda$. Then there are open sets U and V in E_λ and B_λ, respectively, such that $cl\left(q_\lambda(E)\right) \times cl\left(r_\lambda(B)\right) \subseteq U \times V \subseteq W$. By property (B2) for $\mathbf{q}$ and $\mathbf{r}$, there are $\lambda' \geqslant \lambda$ and $\lambda'' \geqslant \lambda$ such that $q_{\lambda\lambda'}(E_{\lambda'}) \subseteq U$ and $r_{\lambda\lambda''}(B_{\lambda''}) \subseteq V$. Then for $\lambda''' \geqslant \lambda', \lambda''$ one has

$$(q_{\lambda\lambda'''} \times r_{\lambda\lambda'''})\,(E_{\lambda'''} \times B_{\lambda'''}) \subseteq U \times V \subseteq W.$$

The next lemma states that the *AHLP* for level maps of compact polyhedral (and thus, compact *ANR*) systems can be expressed in the „$\varepsilon - \delta$ language“.

Lemma 3.9. *Let* $\mathbf{p}=(p_\lambda, 1_\Lambda):\mathbf{E}=(E_\lambda, q_{\lambda\lambda'}, \Lambda)\to\mathbf{B}=(B_\lambda, r_{\lambda\lambda'}, \Lambda)$ *be a level map of compact polyhedral systems. Then,* $\mathbf{p}$ *has the AHLP with respect to a class of spaces* $\mathcal{X}$ *if and only if for every* $\varepsilon>0$ *and every* $\lambda\in\Lambda$ *there is a* $\lambda'\geqslant\lambda$ *and a* $\delta>0$ *such that for each* $X\in\mathcal{X}$ *and any two maps* $h:X\to E_{\lambda'}$, $H:X\times I\to B_{\lambda'}$ *with*

$$d(p_{\lambda'}h, H_0)<\delta, \tag{3}$$

there is a homotopy $\tilde{H}:X\times I\to E_\lambda$ *such that*

$$d(q_{\lambda\lambda'}h, \tilde{H}_0)<\varepsilon \tag{4}$$

$$d(p_\lambda\tilde{H}, r_{\lambda\lambda'}H)<\varepsilon. \tag{5}$$

(λ' and δ are called lifting index and lifting mesh for (λ, ε) respectively).

Proof. *Necessity.* Let $\mathbf{p}$ have the *AHLP* with respect to $\mathcal{X}$ and let $\varepsilon>0$ and $\lambda\in\Lambda$. Then $\mathcal{U}=\left\{B\left(x, \frac{\varepsilon}{2}\right):x\in E_\lambda\right\}$ and $\mathcal{V}=\left\{B\left(x, \frac{\varepsilon}{2}\right):x\in B_\lambda\right\}$ are open coverings of E_λ and B_λ respectively. $\left(\text{Here } B\left(x, \frac{\varepsilon}{2}\right)\text{ denotes the open ball with centre } x \text{ and radius } \frac{\varepsilon}{2}\right)$. Let λ' be a lifting index and let an open covering $\mathcal{V}'$ of $B_{\lambda'}$ be a lifting mesh for λ, $\mathcal{U}$ and $\mathcal{V}$. Let $\delta>0$ be a Lebesgue number of the covering $\mathcal{V}'$ ([3], XI, Theorem 4.5). We claim that λ' and δ are a lifting index and a lifting mesh for (λ, ε). Indeed, let $X\in\mathcal{X}$ and $h:X\to E_{\lambda'}$, $H:X\times I\to B_{\lambda'}$ be maps with $d(p_{\lambda'}h, H_0)<\delta$. By the choice of δ, this means that $(p_{\lambda'}h, H_0)\leqslant\mathcal{V}'$. By the choice of λ' and $\mathcal{V}'$, it follows that there is a homotopy $\tilde{H}:X\times I\to E_\lambda$ such that $(q_{\lambda\lambda'}h, \tilde{H}_0)\leqslant\mathcal{U}$ and $(p_\lambda\tilde{H}, r_{\lambda\lambda'}H)\leqslant\mathcal{V}$. Since $\mathcal{U}$ and $\mathcal{V}$ are ε-coverings, we conclude that

$$d(q_{\lambda\lambda'}h, \tilde{H}_0)<\varepsilon \text{ and } d(p_\lambda\tilde{H}, r_{\lambda\lambda'}H)<\varepsilon.$$

Sufficiency. Let $\lambda\in\Lambda$ and let $\mathcal{U}$ and $\mathcal{V}$ be open coverings of E_λ and B_λ respectively. We put $\varepsilon=\min\{\varepsilon_1, \varepsilon_2\}$, where ε_2, ε_1 are Lebesgue numbers of $\mathcal{U}$ and $\mathcal{V}$ respectively. Let $\lambda'\geqslant\lambda$ and $\delta>0$ be a lifting index and a lifting mesh for (λ, ε). Then it is easily seen that λ' and $\mathcal{V}'=\left\{B\left(x, \frac{\delta}{2}\right):x\in B_{\lambda'}\right\}$ are a lifting index and a lifting mesh for λ, $\mathcal{U}$ and $\mathcal{V}$ (in the sense of (2.10)).

Remark 3.10. Note that Propositions 1 and 2 from [11] remain true if instead of a level map $\mathbf{p}:\mathbf{E}\to\mathbf{B}$ between compact *ANR* sequences we take a level map between compact polyhedral (i.e. compact *ANR*) systems. Therefore, we conclude that the statemant of Lemma 3.9 remains true if $\mathcal{X}$ is the class of all metric spaces and if (3) and (4) are replaced by

$$p_{\lambda'}h=H_0 \tag{3'}$$

$$q_{\lambda\lambda'}h=\tilde{H}_0 \tag{4'}$$

respectively. Moreover, without loss of generality, we can assume $\mathcal{Y}$ to be the class of all topological spaces, because, similarly as in the proof of [13], Theorem 3, we can justify the following statement: If $\mathbf{p}:\mathbf{E}\to\mathbf{B}$ is a level map of compact polyhedral (and thus compact *ANR*) systems, which has the *AHLP* with respect to the class of all metric spaces, then $\mathbf{p}$ has the *AHLP* with respect to the class of all topological spaces.

4. The pull-back of a shape fibration.

We say that a map $p:E\to B$ is *induced* by a level map of inverse systems $\mathbf{p}=(p_\lambda, 1_\Lambda):\mathbf{E}=(E_\lambda, q_{\lambda\lambda'}, \Lambda)\to\mathbf{B}=(B_\lambda, r_{\lambda\lambda'}, \Lambda)$ if $\lim \mathbf{E}=(E, \mathbf{q})$, $\lim \mathbf{B}=(B, \mathbf{r})$ and $p_\lambda q_\lambda = r_\lambda p$ for each $\lambda\in\Lambda$. In that case we write $\lim \mathbf{p}=p$.

Note that from Theorem 3.2 and Theorem 2.8 it follows that every map of compact Hausdorff spaces $p:E\to B$ admits a level map of a compact polyhedral systems $\mathbf{p}:\mathbf{E}\to\mathbf{B}$ which induce it. Also, from Theorem 2.8, Remark 3.10 and Definition 2.11 the next theorem immediately follows:

Theorem 4.1. *A map $p:E\to B$ between compact Hausdoff spaces is a shape fibration if and only if there is a level map of compact polyhedral (compact ANR) systems $\mathbf{p}:\mathbf{E}\to\mathbf{B}$ which induces p and such that it has the AHLP in the sense of Remark* 3.10 *with respect to the class of all topological spaces.*

The next Proposition is needed in the sequel.

Proposition 4.2. *If lim $\mathbf{B}=(B, \mathbf{r})$ and $x, y\in B$, then $r_\lambda(x)=r_\lambda(y)$ for each $\lambda\in\Lambda$ implies $x=y$.*

Proof. Since the inverse limit B is a subset of a product $\prod_\lambda B_\lambda$ and since $r_\lambda:B\to B_\lambda$ are projections ([3], p. 427), $r_\lambda(x)=r_\lambda(y)$ for each $\lambda\in\Lambda$ means that each coordinate x_λ of x is equal to the corresponding coordinate y_λ of a point y, i.e. $x=y$.

Now we can prove the main Theorem of this paper, which generalizes the analogus result for compact metric spaces ([7], Theorem 2.1 or [5], Theorem 3.3).

Theorem 4.3. *Let $p:E\to B$, $p':E'\to B'$, $f:B'\to B$ and $g':E'\to E$ be maps of compact Hausdorff spaces. If (E', p', g') is a pull-back of (B, p, f) and if p is a shape fibration, then p' is also a shape fibration.*

Proof. First notice that, by Theorem 2.8, one can conclude that $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ is a compact level polyhedral (comact *ANR*) reso'ution of a map $p:E\to B$ between compact Hausdorff spaces if and only if $\lim \mathbf{p}=p$. Therefore, by Corollary 3.5, for maps $p:E\to B$ and $f:B'\to B$ there are level maps of compact polyhedral (compact *ANR*) systems $\mathbf{p}=(p_\lambda, 1_\Lambda):\mathbf{E}=(E_\lambda, q_{\lambda\lambda'}, \Lambda)\to\mathbf{B}=(B_\lambda, r_{\lambda\lambda'}, \Lambda)$ and $\mathbf{f}=(f_\lambda, 1_\Lambda):\mathbf{B}'=(B'_\lambda, s_{\lambda\lambda'}, \Lambda)\to\mathbf{B}$ with $\lim \mathbf{p}=p$ and $\lim \mathbf{f}=f$.

Let

$$f^*(E)=\{(x, y)\in B'\times E: f(x)=p(y)\}\subseteq B'\times E;$$

$$\pi: f^*(E)\to B' \text{ -- the first projection;}$$

$$g: f^*(E)\to E \text{ — the second projection;}$$

$$f_\lambda^*(E_\lambda)=\{(x, y)\in B'_\lambda\times E_\lambda: f_\lambda(x)=p_\lambda(y)\}\subseteq B'_\lambda\times E_\lambda,\ \lambda\in\Lambda;$$

$$\pi_\lambda: f_\lambda^*(E_\lambda)\to B'_\lambda \text{ — the first projection;}$$

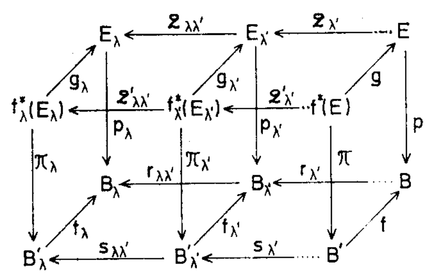
$$g_\lambda: f_\lambda^*(E_\lambda)\to E_\lambda \text{ — the second projection.}$$

Let $q'_\lambda: f^*(E)\to f_\lambda^*(E_\lambda)$ be given by $q'_\lambda=(s_\lambda\pi, q_\lambda g)$, i.e.

$q'_\lambda(x, y)=(s_\lambda(x), q_\lambda(y))$ and $q'_{\lambda\lambda'}: f_{\lambda'}^*(E_{\lambda'})\to f_\lambda^*(E_\lambda)$, $\lambda\leqslant\lambda'$, by

$q'_{\lambda\lambda'}=(s_{\lambda\lambda'}\pi_{\lambda'}, q_{\lambda\lambda'}g_{\lambda'})$, i.e. $q'_{\lambda\lambda'}(x, y)=(s_{\lambda\lambda'}(x), q_{\lambda\lambda'}(y))$.

(See the diagram below).



It is well known that $(f^*(E), \pi, g)$ is a pull-back for (B, p, f). It is known also that the pull-back in each category is determined uniquely up to an isomorphism in that category ([6], p. 60). Hence, $f^*(E)$ and E' are homeomorphic spaces, and thus, we can identify E' with $f^*(E)$, $p': E'\to B'$ with $\pi: f^*(E)\to B'$ and $g': E'\to E$ with $g: f^*(E)\to E$. Therefore, it is sufficient to show that π is a shape fibration.

From the above diagram it is clear that $\mathbf{f}^*(\mathbf{E})=(f_\lambda^*(E_\lambda), q'_{\lambda\lambda'}, \Lambda)$ is an inverse system. We see that

$$\text{(1)} \qquad \lim \mathbf{f}^*(\mathbf{E})=(f^*(E), \mathbf{q}').$$

Indeed, $(x, y)\in f^*(E)$ implies $f(x)=p(y)$, and thus, $r_\lambda f(x)=r_\lambda p(y)$, i.e. $f_\lambda s_\lambda(x)=p_\lambda q_\lambda(y)$ for each $\lambda\in\Lambda$. This means that $q'_\lambda(x, y)=(s_\lambda(x), q_\lambda(y))\in f_\lambda^*(E_\lambda)$ for each $\lambda\in\Lambda$. Hence, $(x, y)\in\lim \mathbf{f}^*(\mathbf{E})$. Conversely, let $(x, y)\in\lim \mathbf{f}^*(\mathbf{E})$. Then $q'_\lambda(x, y)=(s_\lambda(x), q_\lambda(y))\in f_\lambda^*(E_\lambda)$ for each $\lambda\in\Lambda$, which implies that $f_\lambda s_\lambda(x)=p_\lambda q_\lambda(y)$, i.e. $r_\lambda f(x)=r_\lambda p(y)$ for each $\lambda\in\Lambda$. By Proposition 4.2 this implies $f(x)=p(y)$, i,e. $(x, y)\in f^*(E)$.

Furthermore, from the above comutative diagram it follows that $\pi=(\pi_\lambda, 1_\Lambda):\mathbf{f}^*(\mathbf{E})\to\mathbf{B}'$ is a level map with lim $\pi=\pi$. Since $\mathbf{B}'$ and $\mathbf{E}$ are inverse systems of compact polyhedra and lim $\mathbf{B}'=(B', \mathbf{s})$, lim$\mathbf{E}=(E, \mathbf{q})$ from Proposition 3.8 and Theorem 2.8 we conclude that lim $(\mathbf{B}'\times\mathbf{E})=(B'\times E, \mathbf{s}\times\mathbf{q})$. If we now apply Theorem 3.7 with $B'\times E$ instead of $E\times B, f^*(E)$ instead of B_0 and $\mathbf{f}^*(\mathbf{E})$ instead of $\mathbf{B}_0$ we conclude that tor each $\lambda\in\Lambda$ there is a closed polyhedral neighborhood E''_λ of $f^*_\lambda(E_\lambda)$ in $B'_\lambda\times E_\lambda$ such that

$$q''_{\lambda\lambda'}(E''_{\lambda'})\subseteq \text{Int } E''_\lambda,\ \lambda<\lambda',$$

where $q''_{\lambda\lambda'}=(s_{\lambda\lambda'}\times q_{\lambda\lambda'})\,|\,E''_{\lambda'}$, and

(2) $$\lim \mathbf{E}''=(f^*(E), \mathbf{q}''),$$

where $\mathbf{q}''=(\mathbf{s}\times\mathbf{q})\,|\,f^*(E):f^*(E)\to\mathbf{E}''=(E''_\lambda, q''_{\lambda\lambda'}, \Lambda)$.

There is no loss of generality in assuming that Λ is a cofinite set without maximal element. By this assumption we can always assume that $j(\lambda)>\lambda$ for each $\lambda\in\Lambda$, where $j(\lambda)$ denotes a lifting index for λ. Now using induction on the number of predecessors of λ different from λ, we can assign to each $\lambda\in\Lambda$ a closed polyhedral neighborhood E'_λ of $f^*_\lambda(E_\lambda)$ in $B'_\lambda\times E_\lambda$, positive numbers ε_λ, δ_λ and an index $j(\lambda)\in\Lambda$ such that $j(\lambda)$ is a lifting index and δ_λ a lifting mesh for $(\lambda, \varepsilon_\lambda)$ with respect to $\mathbf{p}$ (in the sense of Remark 3.10) and the following conditions are fulfilled:

(3) $$q'_{\lambda\lambda'}(E'_{\lambda'})\subseteq E'_\lambda,\ \lambda\leqslant\lambda',$$

where $q'_{\lambda\lambda'}(x, y)=(s_{\lambda\lambda'}(x), q_{\lambda\lambda'}(y))$ for each $(x, y)\in E'_{\lambda'}$;

(4) $$f^*_\lambda(E_\lambda)\subseteq E'_\lambda\subseteq E''_\lambda,\ \lambda\in\Lambda;$$

(5) $$d(f_{j(\lambda)}\pi_{j(\lambda)}\,|\,E'_{j(\lambda)}, p_{j(\lambda)}g_{j(\lambda)}\,|\,E'_{j(\lambda)})<\delta_\lambda;$$

(6) $$\text{For } (x, y)\in B'_\lambda\times E_\lambda,\ d(f_\lambda(x), p_\lambda(y))<\varepsilon_\lambda\Rightarrow(x, y)\in E'_\lambda.$$

Let Λ_k be the set of all $\lambda\in\Lambda$ with exactly k predecessors different from λ. If $\lambda\in\Lambda_0$, we put $E'_\lambda=E''_\lambda$. The number ε_λ is determined in this manner: consider the map $h_\lambda:B'_\lambda\times E_\lambda\to R^+\cup\{0\}$ given by

$$h_\lambda(x,y)=d(f_\lambda(x), p_\lambda(y)),\quad (x, y)\in B'_\lambda\times E_\lambda,$$

where d is a metric in B_λ. Then h_λ is a closed map and $h_\lambda^{-1}(0)=f^*_\lambda(E_\lambda)$. Since Int E'_λ is an open neighborhood of $f^*_\lambda(E_\lambda)$ in $B'_\lambda\times E_\lambda$, by [5], Proposition 2.6, there is an open neighborhood of 0 in $R^+\cup\{0\}$, i.e. there is an $\varepsilon_\lambda>0$ such that $h_\lambda^{-1}([0, \varepsilon_\lambda))\subseteq$ Int $E'_\lambda\subseteq E'_\lambda$. It is readily seen that for such an ε_λ (6) holds. Now let $j(\lambda)\in\Lambda$ and $\delta_\lambda>0$ be a litfting index and a lifting mesh for $(\lambda, \varepsilon_\lambda)$. Since $j(\lambda)>\lambda$ (i.e. $j(\lambda)\notin\Lambda_0$) the conditions (3) and (5) need not be verified. Condition (4) is evidently fulfilled.

Now suppose that for each $\lambda \in \bigcup_{m=0}^{k-1} \Lambda_m$ we have already defined closed polyhedral neighborhoods E'_λ, positive numbers ε_λ, δ_λ and indices $j(\lambda) \in \Lambda$ for which (3), (4) and (6) hold, while (5) holds for those $\lambda \in \bigcup_{m=0}^{k-1} \Lambda_m$ for which $j(\lambda) \in \bigcup_{m=0}^{k-1} \Lambda_m$.

Let $\lambda \in \Lambda_k$ and let $\lambda_1, \lambda_2, \ldots, \lambda_k < \lambda$ be all the predecessors of λ. Let

$$P(\lambda) = \{\lambda' \in \Lambda : \lambda' < \lambda, \quad j(\lambda') = \lambda\}.$$

For each $\lambda' \in P(\lambda)$ we put $0_{j(\lambda')} = h_\lambda^{-1}([0, \delta_{\lambda'}))$. Then $0_{j(\lambda')}$ is an open neighborhood of $f_\lambda^*(E_\lambda)$ in $B'_\lambda \times E_\lambda$. Furthermore

$$\operatorname{Int}\left(E''_\lambda \cap \bigcap_{i=1}^{k} q'^{-1}_{\lambda_i \lambda}(E_{\lambda_i}) \cap \bigcap_{\lambda' \in P(\lambda)} 0_{j(\lambda')}\right)$$

is an open neighborhood of a closed set $f_\lambda^*(E_\lambda)$ in $B'_\lambda \times E_\lambda$. Consequently, there is a closed polyhedral neighborhood E'_λ of $f_\lambda^*(E_\lambda)$ in $B_\lambda \times E_\lambda$ such that

$$E'_\lambda \subseteq \operatorname{Int}\left(E''_\lambda \cap \bigcap_{i=1}^{k} q'^{-1}_{\lambda_i \lambda}(E'_{\lambda_i}) \cap \bigcap_{\lambda' \in P(\lambda)} 0_{j(\lambda')}\right) \tag{7}$$

([14], § 6, Lemma 7). The number ε_λ for which (6) holds is defined similarly as in the first stage of the induction. Now, $j(\lambda)$ and δ_λ are taken to be a lifting index and a lifting mesh for $(\lambda, \varepsilon_\lambda)$ respectively. From (7) it is clear that (3) and (4) hold Condition (5) must be verified only for $\lambda' \in P(\lambda)$. Since $\lambda' \in P(\lambda)$ implies $j(\lambda') = \lambda \in \Lambda_k$, by (7) $E'_{j(\lambda')} \subseteq 0_{j(\lambda')} = h_\lambda^{-1}([0, \delta_{\lambda'}))$. This means that for each $(x, y) \in E'_{j(\lambda')}$ one has

$$h_\lambda(x, y) = d(f_\lambda(x), p_\lambda(y)) = d(f_{j(\lambda')}(x), p_{j(\lambda')}(y)) =$$
$$= d(f_{j(\lambda')}\pi_{j(\lambda')}(x, y), p_{j(\lambda')} g_{j(\lambda')}(x, y)) < \delta_{\lambda'},$$

i.e. (5) hold. Thus, the inductive construction of neighborhoods E_λ' is finished.

Now from (1), (2) and (4) it follows

$$\lim \boldsymbol{E}' = (f^*(E), \boldsymbol{q}'), \tag{8}$$

where $\boldsymbol{q}' = (q'_\lambda, \Lambda) : f^*(E) \to \boldsymbol{E}' = (E'_\lambda; q'_{\lambda\lambda'}, \Lambda)$.

For each $\lambda \in \Lambda$ let $\pi'_\lambda : E'_\lambda \to B'_\lambda$ be the projection to the first factor. Then $\pi' = (\pi'_\lambda, 1_\Lambda) : \boldsymbol{E}' \to \boldsymbol{B}'$ is a level map of systems of compact polyhedra (compact *ANR*'s) and $\lim \pi' = \pi : f^*(E) \to B'$. We will show that π' has *HLP* with respect to the class of all topological spaces, and thus Theorem 4.3 will be proved.

We claim that $j(\lambda)$ constructed as above is a lifting index for λ (with respect to π'). Indeed, let X be a topolgical space and let $h : X \to E'_{j(\lambda)}$, $H : X \times I \to B'_{j(\lambda)}$ be maps satisfying

$$H_0 = \pi'_{j(\lambda)} h. \tag{9}$$

Then, for maps $g_{j(\lambda)} h: X \to E_{j(\lambda)}$ and $f_{j(\lambda)} H: X \times I \to B_{j(\lambda)}$ one has (by (5) and (9))

$$d(f_{j(\lambda)} H_0,\ p_{j(\lambda)} g_{j(\lambda)} h) < \delta_\lambda.$$

Since $j(\lambda)$ is a lifting index and δ_λ is a lifting mesh for $(\lambda.\ \varepsilon_\lambda)$ (with respect to **p**), we conclude that that there is a homotopy $\tilde{H}: X \times I \to E_\lambda$ such that

$$q_{\lambda j(\lambda)} g_{j(\lambda)} h = \tilde{H}_0, \tag{10}$$

$$d(p_\lambda \tilde{H},\ r_{\lambda j(\lambda)} f_{j(\lambda)} H) < \varepsilon_\lambda. \tag{11}$$

Now define $\hat{H}: X \times I \to E'_\lambda$ by

$$\hat{H}(x, t) = (s_{\lambda j(\lambda)} H(x, t),\ \tilde{H}(x, t)),\ (x, t) \in X \times I. \tag{12}$$

Since for each $(x, t) \in X \times I$,

$$d(f_\lambda s_{\lambda j(\lambda)} H(x, t),\ p_\lambda \tilde{H}(x, t)) = d(p_\lambda \tilde{H}(x, t),\ r_{\lambda j(\lambda)} f_{j(\lambda)} H(x, t)) < \varepsilon_\lambda,$$

(6) implies

$$\hat{H}(x, t) = (s_{\lambda j(\lambda)} H(x, t),\ \tilde{H}(x, t)) \in E'_\lambda.$$

Hence, $\hat{H}$ is well defined. Furthermore, by (12), (9) and (10) we obtain

$$\hat{H}(x, 0) = (s_{\lambda j(\lambda)} \pi'_{j(\lambda)} h(x),\ q_{\lambda j(\lambda)} g_{j(\lambda)} h(x)) = q'_{\lambda j(\lambda)} h(x),$$

i.e. $\hat{H}_0 = q'_{\lambda j(\lambda)} h$. Also, by (12), we have $\pi'_\lambda \hat{H} = s_{\lambda j(\lambda)} H$.

Hence, Theorem 4.3 is proved.

If in the proof of Theorem 4.3 we take identity $1_B: B \to B$ instead of $f: B' \to B$, then that proof shows that the following theorem holds:

Theorem 4.4. *Let* $\mathbf{p} = (p_\lambda, 1_\Lambda): (E_\lambda, q_{\lambda\lambda'}, \Lambda) \to \mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$ *be a level map of compact polyhedral systems which induces a map* $p: E \to B$ *between compact Hausdorff spaces. If* **p** *has the AHLP with respect to the class of all topological spaces* $\mathcal{X}$ *(in the sense of Remark* 3.10), *then there is a level map* $\mathbf{p}': \mathbf{E} \to \mathbf{B}$ *of compact polyhedral systems which induces the same map* p *and has the HLP with respect to* $\mathcal{X}$.

Theorem 4.4 is a generalization of [11], Theorem 2.

From Theorem 4.1 and Theorem 4.4 we obtain the following theorem.

Theorem 4.5. *A map* $p: E \to B$ *between compact Hausdorff spaces is a shape fibration if and only if there is a level map of compact polyhedral systems* $\mathbf{p}: \mathbf{E} \to \mathbf{B}$ *which induces* p *and has the HLP with respect to the class of all topological spaces.*

5. The shape equivalence of fibers

In this section we show the following theorem:

Theorem 5.1. *Let $p: E \to B$ be a shape fibration between compact Hausdorff spaces. If $x, y \in B$ are points which can be joined by a path in B, then $Sh(X) = Sh(Y)$, where $X = p^{-1}(x)$, $Y = p^{-1}(y)$.*

Note that this theorem is a generalization of [11], Theorem 3. The proof is patterned after the proof of [11], Theorem 3.

Since $p: E \to B$ is a shape fibration of compact Hausdorff spaces, by Theorem 4.5, there is a level map of compact polyhedral systems $\mathbf{p} = (p_\lambda, 1_\Lambda): \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda) \to \mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$ with $\lim \mathbf{p} = p$, which has the *HLP* with respect to the class of all topological spaces. Without loss of generality we can assume that Λ is a cofinite set. Let $\omega: I \to B$ be a path in B with $\omega(0) = x$, $\omega(0) = y$. For each $\lambda \in \Lambda$ we put $\omega_\lambda = r_\lambda \omega: I \to B_\lambda$, $x_\lambda = r_\lambda(x)$, $y_\lambda = r_\lambda(y)$, $X_\lambda = p_\lambda^{-1}(x_\lambda)$, $Y_\lambda = p_\lambda^{-1}(y_\lambda)$. Then ω_λ is a path in B_λ connecting x_λ with y_λ. We obtain, thus, a map of systems $\mathbf{q} | X: X \to \mathbf{X} = (X_\lambda, q_{\lambda\lambda'} | X_{\lambda'}, \Lambda)$ and $\mathbf{q} | Y: Y \to \mathbf{Y} = (Y_\lambda, q_{\lambda\lambda'} | Y_{\lambda'}, \Lambda)$ Since $p: E \to B$ is a closed map and X_λ, Y_λ, X and Y are compact Hausdorff spaces, it follows from Theorem 2.8 and [5], Theorem 2.5 that

$$\lim \mathbf{X} = (X, \mathbf{q} | X) \text{ and } \lim \mathbf{Y} = (Y, \mathbf{q} | Y).$$

By a *tracing* of a path $\omega: I \to B$ we mean an increasing function $g: \Lambda \to \Lambda$ together with a family of homotopies $G_\lambda: X_{g(\lambda)} \times I \to E_\lambda$ of the form

$$G_\lambda = g_{\lambda\bar{\lambda}} \tilde{H}_{\bar{\lambda} g(\lambda)}, \tag{1}$$

where $\bar{\lambda}$ is the lifting index for λ such that $g(\lambda) \leqslant \bar{\lambda}$, and $\tilde{H}_{\bar{\lambda} g(\lambda)}: X_{g(\lambda)} \times I \to E_{\bar{\lambda}}$ is a homotopy satisfying

$$\tilde{H}_{\bar{\lambda} g(\lambda)}(x, 0) = q_{\bar{\lambda} g(\lambda)}(x) \tag{2}$$

$$p_{\bar{\lambda}} \tilde{H}_{\bar{\lambda} g(\lambda)}(x, t) = \omega_{\bar{\lambda}}(t), \tag{3}$$

for all $t \in I$, $x \in X_{g(\lambda)}$. We denote this tracing of ω by (g, G_λ).

Lemma 5.2. *Every path $\omega: I \to B$ admits a tracing (g, G_λ).*

Proof. Let $\bar{\lambda}$ be a lifting index for $\lambda \in \Lambda$ and let $g': \Lambda \to \Lambda$ be the function which assigns to each $\lambda \in \Lambda$ a lifting index $g'(\lambda)$ for $\bar{\lambda}$. Since Λ is cofinite, there is an increasing $g: \Lambda \to \Lambda$ with $g(\lambda) \geqslant g'(\lambda)$ for each $\lambda \in \Lambda$ ([9], Lemma 2, p. 1.7). Since $g'(\lambda)$ is a lifting index for $\bar{\lambda}$, we conclude that $g(\lambda)$ is also a lifting index for λ. Thus, $g(\lambda) \geqslant \lambda$. Let $H: X_{g(\lambda)} \times I \to B_{g(\lambda)}$ be given by $H(x, t) = \omega_{g(\lambda)}(t)$, and let $h: X_{g(\lambda)} \to E_{g(\lambda)}$ be the inclusion map. Then $H_0 = p_{g(\lambda)} h$. By the choice of $g(\lambda)$, we conclude that there is a homotopy $\tilde{H}_{\bar{\lambda} g(\lambda)}: X_{g(\lambda)} \times I \to E_{\bar{\lambda}}$ satisfying (2) and (3). We define $G_\lambda: X_{g(\lambda)} \times I \to E$ by (1). Hence, Lemm 5.2 is proved.

Similarly as Lemmas 3, 4, 5, 6, 7, of [11] we can prove the following lemmas.

Lemma 5.3. *Let* (g, G_λ) *be a tracing of* ω *and let* $g_\lambda: X_{g(\lambda)} \to Y_\lambda$, $\lambda \in \Lambda$, *be defined by*

$$g_\lambda(x) = G_\lambda(x, 1), \quad x \in X_{g(\lambda)}.$$

Then for all $\lambda \leqslant \lambda'$ *one has*

$$q_{\lambda\lambda'} g_{\lambda'} \simeq g_\lambda q_{g(\lambda) g(\lambda')}.$$

Lemma 5.4 *There is a unique shape morphism* $\mathbf{g}: X \to Y$ *such that* $S([g_\lambda q_{g(\lambda)} | X]) = S([q_\lambda | Y])\,\mathbf{g}$, *where* S *is a shape functor* ([9], *p.* 2.6).

(The tracing (g, G_λ) of ω is said to induce $\mathbf{g}$).

Lemma 5.5. *If* $\omega \simeq \omega'$ *(re'* $\{0, 1\}$*) are paths from x to y and* $\mathbf{g}, \mathbf{g}': X \to Y$ *are shape morphisms induced by tracings* (g, G_λ) *and* (g', G'_λ) *of* ω *and* ω' *respectively, then* $\mathbf{g} = \mathbf{g}'$.

(In view of Lemma 5.5, we say that the shape morphism $\mathbf{g}: X \to Y$ is induced by $[\omega]$).

Lemma 5.6. *If* ω *is a constant path at* $x \in B$, *then* $[\omega]$ *induces the identity shape morphism* 1_X on X.

Lemma 5.7. *If* ω *is a path from x to y,* ω' *is a path from y to z,* $\mathbf{g}: X \to Y$ *is the shape morphism induced by* $[\omega]$ *and* $\mathbf{g}': Y \to Z = p^{-1}(z)$ *is the shape morphism induced by* $[\omega']$, *then* $\mathbf{g}'\mathbf{g}: X \to Z$ *is the shape morphism induced by* $[\omega * \omega']$.

From the above lemmas the proof of Theorem 5.1 follows just as the proof Theorem 3 of [11].

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