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A NOTE ON ANALOGIES BETWEEN THE CHARACTERISTIC AND THE MATCHING POLYNOMIAL OF A GRAPH

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In the present paper we shall determine some propertices of the zeros of the matching polynomial of a graph, which indicate a deeper parallelism between the theory of the matching polynomial and graph spectral theory.

Let G be a graph with n vertices and m edges and let $p(G, k)$ be the number of distinct selections of k independent edges in $G, k=1, 2, \dots$. In addition, let $p(G, 0)=1$ for all graphs G. Then the polynomial

$$\alpha(G)=\alpha(G, x)=\sum_{k=0}^{m}(-1)^k p(G, k)\, x^{n-2k} \tag{1}$$

is called the matching polynomial of the graph G. For a recent review on $\alpha(G)$ see [6], where also the applications of this polynomial in several physical and chemical theories are reported. In the mathematical literature $\alpha(G)$ was first considered by Farrell [2] and, independently, by the present author [7] (see also [3]). In [7] $\alpha(G)$ was named the acyclic polynomial.

The characteristic polynomial of the adjacency matrix of the graph G is called the characteristic polynomial of G and will be denoted by $\Phi(G)=\Phi(G, x)$. Some of its well known properties are the following (see for example [1], pp. 17—22).

1a. All the zeros of $\Phi(G)$ are real.

Let $x_i(G)$, $i=1, \dots, n$ be the zeros of $\Phi(G)$ and let us adopt the convention $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$. The largest zero of $\Phi(G)$ is called [1] the index of the graph G and will be denoted by $X(G)$.

2a. If v is an arbitrary vertex of G, then for $i=1, \dots, n-1$,

$$x_i(G) \geqslant x_i(G-v) \geqslant x_{i+1}(G).$$

In particular, $X(G) \geqslant X(G-v)$.

3a. If G is connected, then $X(G) > X(G-v)$.

4a. If G is connected, then $X(G) > x_j(G)$, $j=2, \dots, n$.

5a. If e is an arbitrary edge of G, then $X(G) \geqslant X(G-e)$. If G is connected, then $X(G) > X(G-e)$.

6a. If H is *a* subgraph of G, then $X(G) \geqslant X(H)$. If G is connected, then $X(G) > X(H)$.

7a. If G is connected, then $X(P_n) \leqslant X(G) \leqslant X(K_n)$, with P_n and K_n denoting the path and the complete graph on n vertices.

The properties 1a — 7a can be established immediately by using several classical theorems of matrix theory. Since the definition of the matching polynomial is combinatorial in nature, the methods of linear algebra are by no means applicable in the study of the properties of the zeros of $\alpha(G)$. Nevertheless it can be shown that $\Phi(G)$ and $\alpha(G)$ possess a number of analogous properties. In particular, in the present paper we would like to point out the statements 1b—7b, concerning the zeros of $\alpha(G)$, which fully parallel the properties 1a—7a of the zeros of $\Phi(G)$.

Proposition 1b. *All the zeros of* $\alpha(G)$ are real.

Let $y_i(G)$, $i=1, \ldots, n$ be the zeros of $\alpha(G)$ and let us adopt the convention $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$. The largest zero of $\alpha(G)$ will be called the matching index of the graph G and will be denoted by $Y(G)$.

Proposition 2b. *If v is an arbitrary vertex of G, then for $i=1, \ldots, n-1$, $y_i(G) \geqslant y_i(G-v) \geqslant y_{i+1}(G)$. In particular, $Y(G) \geqslant Y(G-v)$.*

Proposition 3b. *If G is connected, then $Y(G) > Y(G-v)$.*

Proposition 4b. *If G is connected, then $Y(G) > y_j(G)$, $j=2, \ldots, n$.*

Proposition 5b. *If e is an arbitrary edge of G, then $Y(G) \geqslant Y(G-e)$. If G is connected, then $Y(G) > Y(G-e)$.*

Proposition 6b. *If H is a subgraph of G, then $Y(G) \geqslant Y(H)$. If G is connected, then $Y(G) > Y(H)$.*

Propisition 7b. *If G is connected, then*

$$Y(P_n) \leqslant Y(G) \leqslant Y(K_n), \tag{2}$$

with P_n and K_n being the same graphs as in 7a.

Some further relations between $\alpha(G)$ and $\Phi(G)$ which are worth mentioning are the following [6].

8. $\alpha(G) = \Phi(G)$ if and only if G is a forest.

9. Let C be a regular graph of degree two with $p(C)$ components. Then

$$\Phi(G) = \alpha(G) + \sum_C (-2)^{p(C)} \alpha(G-C), \quad \alpha(G) = \Phi(G) + \sum_C (+2)^{p(C)} \Phi(G-C),$$

with the summation going over all regular graphs of degree two which are as subgraphs contained in G.

10. $Y(G) \leqslant X(G)$. If G is connected, then the equality sign holds if and only if G is a tree.

11. For every graph G there exists a forest $F=F(G)$, such that $\alpha(G)$ is a divisor of $\Phi(F)$.

The results 1b and 2b are already known in the literature [8] and several proofs of them are nowadays offered [4, 5, 6, 8]. (Note also that 1b follows immediately from 1a and 11.)

The Propositions 3b—7b are reported here for the first time.

Proof of Proposition 3b. First of all note that if 3b is true, then from 2b we immediately deduce 4b. In order to prove 3b we shall use an induction argument.

For all connected graphs with two and three vertices we can readily check the validity of Proposition 3b. Suppose that 3b, and therefore also 4b, hold for all graphs with less than n vertices.

Let the vertex v of the graph G be adjacent to the vertices $w_1, w_2, \ldots, w_d$. Then the following recursion relation holds for the matching polynomial of G [6],

(3) $$\alpha(G) = x\,\alpha(G-v) - \sum_{j=1}^{d} \alpha(G-v-w_j).$$

Let the matching index of $G-v$ has algebraic multiplicity b ($b \geqslant 1$). Let $G-v$ be composed of the components H_i, $i=1, \ldots, t$ ($t \geqslant b$). Then [6]

(4) $$\alpha(G-v) = \prod_{j=1}^{t} \alpha(H_j).$$

The components H_i have, of course, less than n vertices. Every H_i, $i=1, \ldots, t$ is a connected graph. Then on the basis of the induction hypothesis the matching index of every H_i, $i=1, \ldots, t$ has algebraic multiplicity one. From eq. (4) is then evident that the matching index of exactly b components H_i is equal to the matching index of $G-v$. Let these be the components H_i, $i=1, \ldots, b$.

If the vertex w_j does not belong to any of H_i, $i=1, \ldots, b$, then $G-v-w_j$ contains all the components H_i, $i=1, \ldots, b$ and $Y(G-v)$ is the matching index of $G-v-w_j$ and its algebraic multiplicity (with respect to $\alpha(G-v-w_j)$) is b.

If, on the other hand, the vertex w_j belongs to one of the components H_i, $i=1, \ldots, b$, then according to the induction hypothesis $Y(G-v)$ is a zero of the polynomial $\alpha(G-v-w_j)$ with algebraic multiplicity $b-1$.

Consequently, $Y(G-v)$ is a zero of the polynomial $\sum_{j=1}^{d} \alpha(G-v-w_j)$ with algebraic multiplicity $b-1$. Because of eq. (3), $Y(G-v)$ is also a zero of $\alpha(G)$ with algebraic multiplicity $b-1$. (Of course if $b=1$, then $Y(G-v)$ is neither a zero of $\alpha(G)$ nor of $\sum_{j=1}^{d} \alpha(G-v-w_j)$.)

Two cases are to be distinguished now. If $b=1$, then by setting $x = Y(G-v)$ in eq. (3) we obtain $\alpha(G, x) < 0$, from which the inequality $Y(G) > Y(G-v)$ follows.

If $b>1$, then we differentiate eq. (3) $b-1$ times with respect to the variable x. Then by setting $x = Y(G-v)$ we obtain

(5) $$(d^{b-1}/dx^{b-1})\,\alpha(G, x) < 0,$$

since $(d^{b-1}/dx^{b-1})\,\alpha(G-v-w_j, x)$ must be non-negative for all $j=1, \ldots, d$ and must be positive for at least one j. From (5) we immediately conclude that $Y(G-v)$ is not the matching index of G and therefore the inequality $Y(G) > Y(G-v)$ must hold.

This completes the proof of Proposition 3b.

P r o o f o f P r o p o s i t i o n 4b. Combine 2b and 3b.

P r o o f o f P r o p o s i t i o n 5b. Let the edge e connect the vertices v and w. Then [6]

$$\alpha(G)=\alpha(G-e)-\alpha(G-v-w). \tag{6}$$

Since $G-v-w=(G-e)-v-w$, from Proposition 2b we have the inequality

$$Y(G-e)\geqslant Y(G-v-w), \tag{7}$$

Therefore $\alpha(G-v-w,\ Y(G-e))\geqslant 0$. Consequently, by setting $x=Y(G-e)$ in the eq. (6) we get $\alpha(G)\leqslant 0$, from which

$$Y(G)\geqslant Y(G-e) \tag{8}$$

follows immediately. Note that if the inequality (7) is strict, then also (8) is strict.

Hence in order to complete the proof we have to demonstrate that if G is connected then (7) is a strict inequality. If G is connected then $G-e$ is either connected or composed of exactly two disconnected, parts, say H_1 and H_2.

In the first case, because of 2b and 3b, $Y(G-e)>Y(G-v)\geqslant Y(G-v-w)$. If, on the other hand, $G-e$ is disconnected, then $Y(G-e)=\max\{Y(H_1), Y(H_2)\}$ and $Y(G-v-w)=\max\{Y(H_1-v), Y(H_2-w)\}$. Because of 3b, $Y(H_1)>Y(H_1-v)$ and $Y(H_2)>Y(H_2-w)$, which implies again the strict inequality (7).

Proposition 5b has been thus proved.

P r o o f o f P r o p o s i t i o n 6b. Apply 2b, 3b and 5b step-by-step to those vertices and edges of G which are not contained in H.

P r o o f o f P r o p o s i t i o n 7b. If T is a spanning tree of G, then by 6b, $Y(T)\leqslant Y(G)$. On the other hand, because of 8, the index and the matching index of a tree coincide.

It is known [9] that among trees with n vertices, P_n has the smallest index. This proves the left inequaliyt in (2). Of course [1], $Y(P_n)=2\cos\dfrac{\pi}{n+1}$.

The right hand side inequality in (2) follows from 6b and the fact that every graph with n vertices is contained as subgraph in K_n.

Note that $\alpha(K_n)$ is equal to the Hermite polynomial of order n [6] and therefore $Y(K_n)$ is equal to the largest zero of the Hermite polynomial. Since the equation $\alpha(K_n)=0$ is not solvable by radicals [10], no explicit expression exists for the matching index of the complete graph.

It is not difficult to see that equality on the left (resp. right) hand side of (2) is obtained if and only if G is isomorphic to P_n (resp. to K_n).

In the present paper we have shown that the zeros of the matching polynomial possess the same fundamental algebraic properties as the graph spectrum. This applies especially to the matching index. Our results suggest that there may exist a more general connection between the two polynomials, which, however, remains still to be discovered.

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