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ORDER-REVERSING MAPS AND UNIQUE FIXED POINTS IN COMPLETE LATTICES

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In [1] A. Björner has proved the following

T h e o r e m A. *Every nontransposing order-reversing map of a complete lattice into itself has a unique fixed point.*

(An order-reversing self-map $f: P \to P$ of a poset P is nontransposing in case there is no $x \in P$ such that $f^2(x)=x<f(x)$.).

Theorem A is directly implied by the following Theorem B (see [3]).

T h e o r e m B. *Let P be a complete lattice, $g, h: P \to P$ two order-preserving maps such that $g=f_1 \circ f_2$, $h=f_2 \circ f_1$. Then*

(i) $f_1(I(h, P)) \subset I(g, P)$, (ii) $f_2(I(g, P)) \subset I(h, P)$.

($I(f, P)$ denotes the set of fixed points of the map $f: P \to P$).

Taking $f_1=f_2=f$ and f order-reversing we have the following special case of Theorem *D*.

T h e o r e m C. *Let P be a complete lattice and $f: P \to P$ an odrer-reversing map. Then $f(I(f^2, P)) \subset I(f^2, P)$.*

The following theorem, which follows from Tarski's theorem and Theorem C, has as a corollary Theorem 1 and makes it more precise.

T h e o r e m. *Let P be a complete lattice and f an order-reversing self-map of P. Then f is nontransposing if and only if $I(f^2, P)$ is a singleton.*

P r o o f. Suppose that f is nontransposing. Put $m=\min I(f^2, P)$ ($I(f^2, P)$ is a non-empty complete lattice, by Tarski's theorem). From Theorem C it follows that $f(m) \geqslant m$, or, since f is nontransposing, $f(m)=m$. Suppose there exists an $x \in I(f^2, P)$, $x \neq m$. From $x>m$ it follows $f(x) \leqslant f(m)=m$, or $f(x)=m$ (by Theorem *C*). Hence $f^2(x)=f(m)$, or $x=m$; a contradiction. So $I(f^2, P)$ is a singleton.

Suppose now that $I(f^2, P)$ is a singleton and denote by m the unique fixed point of f^2. From $f^2(m)=m$ it follows that $f(m)=f^2(f(m))$, so $f(m)$ is a fixed point of f^2. From the uniqueness of m it follows that $f(m)=m$. So f is nontransposing.

R e m a r k. Several known results are corollaries of our theorem. For example, [2, Theorem 1], [6, Theorem 5]. The former result has a game-theoretic corollary [5, Theorem 1] and it can be seen at once that [5, Theorem 2] follows from our theorem.

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