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ON GRAPHS WHOSE SECOND LARGEST EIGENVALUE DOES NOT EXCEED 1

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Abstract: Graphs with second largest eigenvalue not greater than 1 are partially characterized in this paper. Some properties of these graphs are described.

A.J. Hoffman posed the problem of characterizing graphs with the second largest eigenvalue not greater than 1. We shall partially solve the problem in this paper. As it will be seen, such graphs are, roughly speaking, complements of graphs with the least eigenvalue not smaller than -2. Since the latter graphs have been treated extensively in the literature (see [4], pp. 168—198), the present investigation seems to be of some interest.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$ be the eigenvalues of a graph G and of its complement $\bar{G}$, respectively, both in non-increasing order.

Theorem 1. *For any graph G the following inequalities hold:*

$$\lambda_i + \bar{\lambda}_j \geqslant -1 + n\,\delta_{2,\,i+j}, \tag{1}$$

$$\lambda_{n-i+1} + \bar{\lambda}_{n-j+1} \leqslant -1 + n\,\delta_{n+1,\,i+j}, \tag{2}$$

where $2 \leqslant i+j \leqslant n+1$ and $\delta_{p,q}$ is the Kronecker δ-symbol.

Proof. We shall use the well-known Courant-Weyl inequalities. If X, Y are real symmetric matrices of order n and if $Z = X+Y$, then

$$\lambda_{i+j-1}(Z) \leqslant \lambda_i(X) + \lambda_j(Y), \tag{3}$$

$$\lambda_{n-i-j+2}(Z) \geqslant \lambda_{n-i+1}(X) + \lambda_{n-j+1}(Y), \tag{4}$$

where $2 \leqslant i+j \leqslant n+1$, and $\lambda_k(M)$ denotes the k-th largest eigenvalue of the matrix M.

Let A be the adjacency matrix of the graph G and let $\bar{A}$ be the adjacency matrix of the complement $\bar{G}$ of G. If $X = A$ and $Y = \bar{A}$, then $Z = J - I$, where J is a matrix whose all entries are equal to 1 and I is a unit matrix. Since the largest eigenvalue of $J - I$ is $n-1$ and the remaining eigenvalues are equal to -1, we immediately get (1), (2) from (3), (4).

This completes the proof.

Corollary. *Putting $i = j = 1$ in* (1) *and* (2) *we get the inequality*

$$\lambda_1 + \bar{\lambda}_1 \geqslant n-1,$$

which is noted in the literature (*see* [4], *p.* 112).

T h e o r e m 2. *Let G be a graph with $\lambda_2 \leqslant 1$. Then $\overline{G}$ belongs to one of the following classes:*

a) *the smallest eigenvalue of $\overline{G}$ is greater than or equal to* -2;

b) *exactly one eigenvalue of $\overline{G}$ is smaller than* -2.

In addition, if G belongs to a), *then* $\lambda_2 \leqslant 1$.

P r o o f. Putting $i=2$, $j=n-1$ in (1) we get

$$\lambda_2 + \overline{\lambda}_{n-1} \geqslant -1. \tag{5}$$

Now, $\lambda_2 \leqslant 1$ implies $\overline{\lambda}_{n-1} \leqslant -2$, which proves the first part. Putting $i=n-1$, $j=2$ in (2) we have

$$\lambda_2 + \overline{\lambda}_n \leqslant -1. \tag{6}$$

Hence, $\overline{\lambda}_n \leqslant -2$ implies $\lambda_2 \leqslant 1$.

This completes the proof.

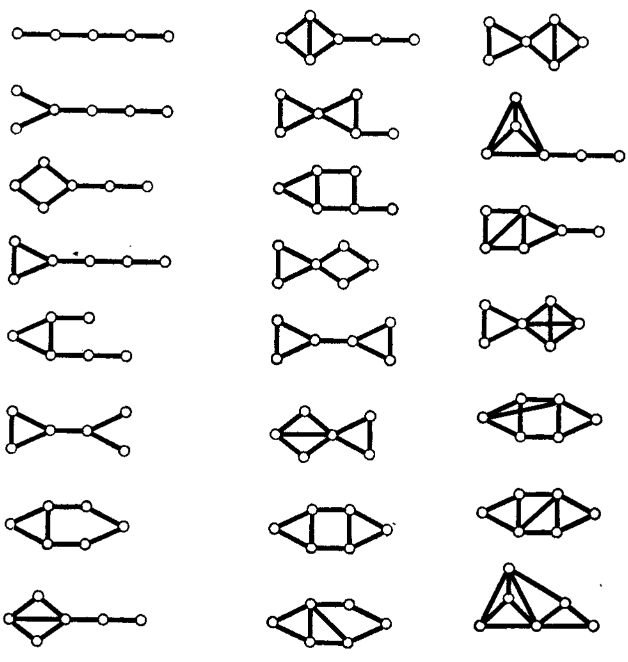

Fig. 1

As it can be seen from the tables of graph spectra from [4], there are no graphs with $\lambda_2 > 1$ among graphs with at most 5 vertices. A list of spectra of 6-vertex graphs, produced by M. Doob and the author by means of a computer, shows that there are exactly 23 out of 112 connected graphs on 6 vertices with $\lambda_2 > 1$. They are displayed in Fig. 1.

All graphs from Fig. 1 are complements of graphs with exactly one eigenvalue smaller than -2. On the other hand, there are graphs with $\lambda_2 \leqslant 1$ which are complements of graphs with exactly one eigenvalue smaller than -2. One such graph is represented in Fig. 2.

According to Theorem 2 if $\overline{G}$ has all eigenvalues greater or equal to -2, then we have $\lambda_2<1$ or $\lambda_2=1$ for G. We shall discuss now when each of these two alternatives occurs.

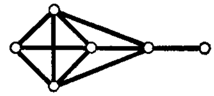

Fig. 2

$L(G)$ denotes the line graph of G. Let $CP(n)=\overline{nK_2}$, be the so-called cocktail party graph. If G has vertices $x_1, x_2, \ldots, x_n$ and if $a_1, a_2, \ldots, a_n$ are non-negative integers, then the generalized line graph $L(G;\ a_1, a_2, \ldots, a_n)$ consists of disjoint copies of $L(G), CP(a_1), \ldots, CP(a_n)$ plus edges joining vertices of $CP(a_i)$ with all vertices of $L(G)$ corresponding to the edges incident with x_i.

Graphs with $\lambda_n \geqslant -2$ are known to be either generalized line graphs or graphs which can be represented by the root system E_8 [1]. The latter graphs will be called exceptional graphs.

Following [2], an eigenvalue of a graph is a main eigenvalue if its eigenspace contains a vector with the coordinate sum different from zero. If λ is an eigenvalue of $\overline{G}$ which is not main, then G contains $-\lambda-1$ as an eigenvalue of the same multiplicity as $\overline{G}$.

T h e o r e m 3. *Complements of graphs, whose smallest eigenvalue is greater than* -2, *have the second largest eigenvalue smaller than* 1.

P r o o f. The inequality $\overline{\lambda}_n > -2$, together with (6), yields $\lambda_2<1$.

This completes the proof.

All graphs with the least eigenvalue greater than -2 have been determined in [7]. According to this paper for a connected graph G with the least eigenvalue greater than -2 one of the following holds:

a) $G=L(T, 1, 0, \ldots, 0)$, where T is a tree,

b) $G=L(H)$, where H is unicyclic with an odd cycle,

c) G is one of 573 graphs that arise from the root system E_8.

C o r o l l a r y. *Complements of graphs, whose all components satisfy a), b), or c), have the second largest eigenvalue less than* 1.

Let now $\overline{G}$ be a graph with the least eigenvalue -2. If the multiplicity of -2 is greater than 1 a non-main eigenvalue of $\overline{G}$ is equal to -2 and $\overline{G}$ has $\lambda_2=1$. If -2 is simple eigenvalue of $\overline{G}$, it will be maped onto $\lambda_2=-1$ of G if and only if it is a non-main eigenvalue.

Therefore we have the following theorem.

T h o r e m 4. *Let G be a graph with the least eigenvalue* -2. *Then G has* $\lambda_2<1$ *if* -2 *is a simple main eigenvalue of G. Otherwise, we have* $\lambda_2=1$.

Using results of [6] we can describe generalized line graph with a simple main eigenvalue -2.

Theorem 5. *Let $G=L(H; a_1, a_2, \ldots, a_n)$ be a connected generalized line graph with the smallest eigenvalue -2. The complement $\overline{G}$ of G has the second largest eigenvalue less than 1 if either H is a tree, $\sum_{i=1}^{n} a_i=2$ and there exist vertices i and j in H which are at an odd distance with $a_i \neq 0$, $a_j \neq 0$ or H contains an odd cycle with $\sum_{i=1}^{n} a_i=1$. Otherwise $\overline{G}$ has the second largest eigenvalue equal to 1.*

Proof. We shall find connected generalized line graphs G with -2 being a simple main eigenvalue. The eigenvalue -2 in line graphs is always a non-main eigenvalue. Therefore we have $\sum_{i=1}^{n} a_i>0$. In that case the multiplicity of -2 is given by $\sum_{i=1}^{n} a_i+m-n$, where m and n are the number of edges and the number of vertices of H. Therefore H is a tree with $\sum_{i=1}^{n} a_i=2$ or H is a unicyclic graph with $\sum_{i=1}^{n} a_i=1$. Now we should construct an eigenvector of -2 with coordinate sum different from 0.

1° H is a tree with $\sum_{i=1}^{n} a_i=2$. Suppose $a_i=2$, other a_j's being equal to 0. Then G contains induced subgraph displayed in Fig. 3. The only (up to a multi plicative constant) eigenvector of G belonging to -2 has coordinates different from 0 only on vertices of the mentioned subgraph. Consequently, the coordinate sum is equal to 0 and -2 is a non-main eigenvalue.

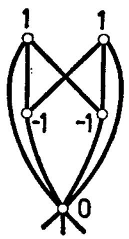

Fig. 3

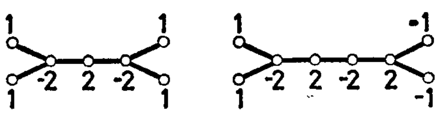

Fig. 4

Suppose now that $a_i=1$, $a_j=1$ and all other a_k's are equal to 0. The subgraph of G induced by vertices with non-zero coordinates of the eigenvector belonging to -2 is displayed in Fig. 4 for two cases (when the distance between vertices i and j in H is 3 and 4). Obviously, the coordinates add up to zero when this distance is even and the coordinate sum is different from zero if the distance is odd.

2° H is unicyclic with $\sum_{i=1}^{n} a_i=1$. The characteristic subgraph depending on whether the cycle is of an even or of an odd length and depending on whether the vertex i of H with $a_i=1$ lies on or outside the cycle, is displayed in Fig. 5. Hence, the cycle must be of an odd length.

This completes the proof.

Let now G be an exceptional graph with the least eigenvalue -2. As earlier, we have $\lambda_2 \leqslant 1$ for G. We have $\lambda_2 < 1$ if and only if -2 is a simple main eigen-

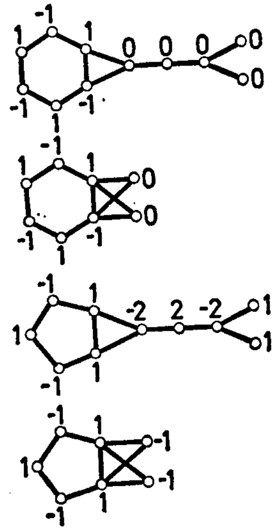

Fig. 5

value of G. A connected exceptional graph with a simple eigenvalue -2 has 7,8 or 9 vertices. Hence, there is a finite number of such graphs. It remains to establish which of them have -2 as a main eigenvalue.

Concerning graph G with $\bar{G}$ having exactly one eigenvalue less than -2, let us note the following observations:

1. If $\lambda_2 < 1$, then $\bar{\lambda}_{n-1} > -2$.

2. If $\bar{\lambda}_n$ is not a main eigenvalue, then $\lambda_2 > 1$.

We conclude with the next theorem showing some properties of graphs with $\lambda_2 \leqslant 1$.

Theorem 6. *If G is a graph with $\lambda_2 \leqslant 1$, then the girth of G is at most* 6 *or G is a forest and the diameter of G is at most* 4.

Proof. A circuit of length more than 6 and a path of length more than 4 have at least two eigenvalues greater than 1. By the interlacing theorem G cannot contain such graphs as induced subgraphs.

This completes the proof.

Upper bounds for the diameter and for the girth mentioned in Theorem 6, have been derived in [5] for complements of line graphs using forbidden subgraph techniques. Now they are extended to all graphs with $\lambda_2 \leqslant 1$ including, of course, generalized line graphs and exceptional graphs.

Graph spectra can be used in solving graph equations (see, for example, [3] where this is recommended as one possible direction of further investigations of graph spectra). The strong restriction, that complements of line graphs have $\lambda_2 \leqslant 1$, enables immediately to solve the following graph equations:

$$L(G) = G_1 \times G_2, \tag{7}$$

$$L(G) = G_1 + G_2, \tag{8}$$

$$L(G) = G_1 * G_2, \tag{9}$$

where $L(G)$ denotes the line graph of G and symbols $\times$, $+$, $*$ denote product, sum and strong product of graphs respectively.

It is well known that eigenvalues of $G_1 \times G_2$ are all possible products $\lambda_i \mu_j$ where λ_i is an eigenvalue of G_1 and μ_j an eigenvalue of G_2. If G_1 and G_2, are not trivial and not totally disconnected, then all solutions of equation (7) are given by $G = K_{m,n}$, $G_1 = K_m$, $G_2 = K_n$ for if G_1 or G_2 would not be complete, $G_1 \times G_2$ would have two eigenvalues greater than 1.

Equations (8) and (9) can be treated similarly.

Equations (7)—(9) have been completely solved by non-spectral means [9].

Some characterizations of graphs with the second largest eigenvalue bounded from above have been given in [8]. It would be interesting to find connections between [8] and results from this paper.

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