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THE SPECTRUM OF LINE GRAPHS OF SOME INFINITE GRAPHS

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Abstract. We introduce the notion of D-spectrum of an infinite graph, we prove some of its properties, and in particu'ar we prove an exstension to the infinite case of a theorem by M. Doob concering the value -2 in the spectrum of line graph of finite graphs. In addition, we determine the D-spectrum of line graphs of almost all complete k-partite and infinite graphs.

0. Introduction

Our original aim was to determine the usual spectrum (as defined in [4]) of the line graph of some complete multipartite infinite graphs in some simple cases, assuming that suitable labellings of their vertex sets are chosen.

But we immediately observed that this desire encounters with great difficulties. That motivated us to introduce a second kind of spectrum, the so-called D-spectrum, which is more easy to determine in these situations.

This notion is, also, much more adopted to line graphs because now we can attain the value $\lambda=-2$ in the spectrum of line graphs. It is not possible for the usual spectrum because $\lambda>-2$ for any eigenvalue λ of $L(G)$, as was proved in [4].

Throughout the paper, we adopt all definitions, notations and results concerning spectra of infinite graphs from our earlier papers [4] and [5].

By an infinite graph, we always mean a denumerable connected (undirected) infinite graph, without loops or multiple edges, whose vertex set $V(G)$ is labelled by the set N of natural numbers.

Its adjacency matrix $A=[a_{ij}]$ is an infinite $N\times N$ matrix where

$$a_{ij}=\begin{cases} a^{i+j-2}, & \text{if } i, j \text{ are adjacent]} \\ 0, & \text{otherwise}\end{cases}$$

and a is a fixed positive constant ($0<a<1$).

The spectrum $\sigma(G)$ of a graph G is the spectrum of this infinite matrix, regarded as a symmetric Hilbert-Schmidt operator in a corresponding Hilbert space H, with a fixed orthonormal basis $\{e_i\}$ ($i\in N$).

The spectrum $\sigma(G)$ of a graph G is always real and it consists of a countable sequence of eigenvalues and the zero.

For any connected infinite graph G we define the D-spectrum of G as the spectrum of the eigenvalue problem $Ax=\lambda Dx$, with $D=\mathrm{diag}\,(1, a^2, a^4, \ldots)$. This spectrum is denoted by $\sigma_D(G)$.

Since the multiplicity is not essentional for our aims, we discuss only the set of all *distinct* D-eigenvalues of graph. We only note the fact that the multiplicity of an eigenvalue $\lambda\in\sigma_D(G)$ can be infinite, too.

1. General properties of D-spectrum of an infinite graph

Since D is a non-negative operator, the D-spectrum is always real, but as the example of an infinite path shows, it can be empty. Moreover, it need not be bounded.

Obviously, the D-spectrum of a graph G is determined by the system of equations

$$(1) \qquad \sum_{j=1}^{\infty} b_{ij} y_j = \lambda y_i \qquad (i\in N),$$

where $b_{ij}=1$ if i,j are adjacent and 0 — otherwise, and $y_i=a^i x_i$ $(i\in N)$ is the "weight" of x at vertex i. Evidently, a vector $\{y_i\}$ which satisfies (1) is an eigenvector if and only if

$$\sum_{i=1}^{\infty} y_i^2/a^{2i} = \sum_{i=1}^{\infty} x_i^2 = \|x\|^2 < \infty .$$

Examples. (1°) If G is an one-sided infinite path, then $\sigma_D(G)=\varnothing$.

(2°) If G is complete infinite graph K_∞, then it can be shown that $\sigma_D(G)=\{-1\}$.

(3°) If G is a complete bipartite infinite graph $K(N_1, N_2)$, then it can be shown that $\sigma_D(G)=\{0\}$

(4°) Let $G=K(N_1, N_2, N_3)$ be complete tripartite infinite graph. Then it can be proved that $\sigma_D(G)=\{0\}$ if at most one of N_i is finite or N_1, N_2 are finite but $|N_1|\neq|N_2|$, and $\sigma_D(G)=\{0, -n\}$ if $|N_1|=|N_2|=n<\infty$ (thus N_3 is infinite).

The example of an infinite path shows that the D-spectrum of a line graph can be empty, too.

An important question is whether D-spetrum of a graph G is invariant under relabellings of its vertex set. In a general case, we conjecture that it is not, but so far we have not come across such an example. So we have:

Problem. *Does there exist any infinite graph G whose D-spectrum is dependent on labellings of its vertex set?*

Neverthless, we can note a characteristic case when $\sigma_D(G)$ is invariant under relabellings of $V(G)$.

P r o p o s i t i o n 1. *If in each labelling of the vertex set $V(G)$ of a graph G, a finite (i.e. with finitely many non-zero coordinates) eigenvector $x \neq 0$ corresponds to each $\lambda \in \sigma_D(G)$, then the spectrum $\sigma_D(G)$ of G is invariant, under relabellings of $V(G)$.*

P r o o f. If $Ax = \lambda Dx$ ($x = \Sigma x_i e_i \neq 0$ is finite) in a labelling of the set $V(G) = N$, then

$$\sum_{j=1}^{\infty} b_{ij} y_j = \lambda y_i \qquad (i \in N),$$

where $y_i = a^i x_i \neq 0$ for at most finitely many $i \in N$.

In any other labelling ω of $V(G)$ (ω is a permutation of N), relation $A_\omega x = \lambda Dx$ takes the form

$$\sum_{i=1}^{\infty} b_{ij} z_{\omega(j)} = \lambda z_{\omega(i)} \qquad (i \in N),$$

where $z_l = y_{\omega^{-1}(l)}$,

But then $z_j \neq 0$ for at most finitely many $j \in N$, so that $\sum_{j=1}^{\infty} z_j^2/a^{2j} < \infty$, which means that $\lambda \in \sigma_D(G_\omega)$. The corresponding eigenvector is then $\{\tilde{x}_i\} = \{z_i/a^i\}$.

Since the converse statement is also true, we get that $\sigma_D(G_\omega) = \sigma_D(G)$. □

Next let G be any infinite graph, and $N_1, N_2, \ldots$ (finitely or infinitely many) the equivalence classes in $V(G)$ with respect to the equivalen cerelation $\sim$ defined by: $x \sim y$ iff vertices x and y are non-adjacent and have the same neighbours.

Then it can be easily checked that for $\lambda \neq 0$, $y_i = \text{const}$ for $i \in N_p$, what in view of condition $\Sigma y_i^2/a^{2i} < \infty$, if N_p is infinite, implies that $y_i = 0$ ($i \in N_p$). Hence, if there are only finitely many finite subsets N_p, then Proposition 1 implies that $\sigma_D(G) \setminus \{0\}$ is invariant under relabellings.

The last observation has the following consequence.

P r o p o s i t i o n 2. *If a graph G has a finite spectrum $\sigma(G)$, then:*

(1°) *$0 \in \sigma_D(G)$ in any labelling of the vertex set;*

(2°) *$\sigma_D(G)$ is invariant under relabellings of its vertex set;*

(3°) *$\sigma_D(G)$ has at most m elements where m is the number of finite subsets N_p which are finite.*

P r o o f. Indeed, if G has a finite spectrum $\sigma(G)$, then as it was proved in [5], $0 \in \sigma(G)$, so that $0 \in \sigma_D(G)$, and G is of "finite type", i.e. there are only finitely many subsets N_p in $V(G)$ (say m), so that $\sigma_D(G)$ is invariant under relabellings.

To prove (3°), let $N_1, \ldots, N_m$ be finite and $N_{m+1}, \ldots, N_k$ be infinite subsets of $V(G)$, and let $\lambda \neq 0$.

Then we easily find that for each N_p, y_i=const ($i \in N_p$), and substituting $v_p = y_i |N_p|$ ($i \in N_p$, $p=1, \ldots, k$), we get that system (1) is equivalent with the finding non-trivial of solutions $v_1, \ldots, v_m$ of the linear system of the type $k \times m$

$$\begin{cases} \sum_{j=1}^{m} \delta_{ij} v_j = \lambda v_i / |N_i| & (i=1, \ldots, m;\ j \neq i) \\ \sum_{j=1}^{m} \delta_{ij} v_j = 0 & (i=m+1, \ldots, k), \end{cases}$$

where $\delta_{ij}=1$ if N_i, N_j are adjacent, and $\delta_{ij}=0$ itherwise.

But since G is connected, we get that $\delta_{ij} \neq 0$ for at least one $i \geqslant m+1$ and $j \leqslant m$, so that an easy argument of linear algebra shows that there are at most $m-1$ real solutions of this system (taking in account their multiplicities, too). This proves (3°).

We note that the bound is achieved in (3°), for instance, in the case of the graph from Example 4°.□

We note that if all characteristic subsets N_p of a graph G are infinite, then immediately $\sigma_D(G) = \varnothing$, so that it is trivially invariant again.

2. *D*-spectrum of line graphs

Concerning the D-spectrum of line graphs, we have at first the following theorem.

Theorem 1. *For any infinite graph G we have $\lambda \geqslant -2$, for arbitrary $\lambda \in \sigma_D(L(G))$, in any labelling of the vertex set of the line graph $L(G)$.*

Proof. As it was proved in [4], we have

$$R'R = A(L(G)) + 2D,$$

where R is the vertex-edge incidence matrix of graph G.

But since $R'R$ is a positive operator, we find

$$\langle A(L(G))x, x\rangle + 2\langle Dx, x\rangle \geqslant 0 \qquad (x \in H).$$

Now $A(L(G))x = \lambda Dx$ implies $(\lambda+2)\langle Dx, x\rangle \geqslant 0$, which, since D is strongly positive operator, means that $\lambda+2 \geqslant 0$.□

The problem which we consider is when $\lambda=-2$ is a D-eigenvalue of the line graph of a graph G. Then we say that G has $(D;-2)$ property.

As in the finite case, it is easy to see that G has $(D;-2)$ property if and only if there is an $x \neq 0$ such that $Rx=0$ ([2, p. 169]).

Moreover, if the indices of G are $v_i=i$ $(i\in N)$ and the vertices of $L(G)$ are f_j $(f_j\in N)$, and if $f_j(i)$ means that the edge f_j is incident with vertex v_i, then $Rx=0$ is equivalent to

$$\sum_{f_j(i)} a^j x_j = 0 \tag{2}$$

for each $v_i\in V(G)$.

We also notice that the last series is always absolutely convergent, so that its sum is independent of the order of its members. Here, $\alpha_j=a^j x_j$ is the weight of x at the edge f_j.

The following theorem extends a result of M. Doob (cf. [2, p. 169] or [1]).

Theorem 2. *If G is a connected infinite graph, then it has* (D; —2) *property if and only if graph G has at least one even cycle or two distinct odd cycles.*

Proof. As in finite case, one part of this claim is trivial ([2, p. 168]). Namely, if G has at least one even cycle or two distinct odd cycles, then easily it has $(D;-2)$ property.

Next we want to prove that if G is a tree or if it has exactly one odd cycle and no even cycle, then it does not have the above mentioned property.

1. Assume at first that G has $(D;\ -2)$ property and that it is a tree.

Then if B is the adjacency matrix of graph $L(G)$ in some fixed labelling of its vertex set, then $Bx=-2\,Dx$ is equivalent to

$$\sum_{f_j(i)} a^j x_j = 0 \qquad (i\in N),$$

or to $\Sigma\alpha_j=0$ for each $i\in N$.

Consider an arbitrary edge $f_j=f_{j_0}$, and denoting $\alpha_{j_0}=\alpha$, prove that $\alpha=0$.

The following elementary inequality will be used in the proof:

Let $\alpha_1, \alpha_2, \dots$ and $p_1, p_2, \dots$ be two finite or infinite sequences of positive numbers such that (if they are infinite) the series $\sum_i \alpha_i$, $\sum_i p_i$ converge. Then:

$$\sum_i \alpha_i^2/p_i \geqslant \Big(\sum_i \alpha_i\Big)^2 \Big/ \Big(\sum_i p_i\Big). \tag{3}$$

Equality holds true in (3) *if and only if $\alpha_i=c\,p_i$ (for every $i\in N$) where* $c>0$.

Now let for every $j\in N$, edge f_j joining vertices $p(j)$ and $q(j)$ of G $(p(j)<q(j))$. Then there is at least one edge $f\in E(G)$ which is incident to a vertex of f_j (say $p(j)$), and let

$$M_1=\{i\in N \mid f_i(p(j)),\ i\neq j\}.$$

Let next M_2 be the set of all indices $i_2 \in N$ where f_{i_2} is adjacent to some f_{i_1} $(i_1 \in M_1,\ i_2 \notin M_1)$ and $i_2 \neq j$.

Similarly, the sets of indices $M_3, M_4, \ldots$ are defined.

Certainly, if some of the sets $M_k (k=2,3,\ldots)$, say M_l is empty, it can easily be concluded that $\alpha_i = 0$ $(i \in M_{l-1})$, upon that $\alpha_i = 0$ $(i \in M_{l-2})$, and so on, finally $\alpha = \alpha_{j_0} = 0$.

Hence, we suppose that all sets $M_1, M_2, \ldots$ are non-empty, and consequently the sequence $\{M_1, M_2, \ldots\}$ is infinite.

Here each of the sets M_i $(i \geqslant 1)$ can be finite or infinite.

To abbreviate the next notations, denote for any $\nu \in N$:

$$S_\nu = \sum_{i \in M_\nu} \alpha_i^2 / \alpha^{2i}, \qquad Q_\nu = \sum_{i \in M_\nu} a^{2i},$$

$$\bar{P}_\nu = \sum_{i \in M_\nu} |\alpha_i|, \qquad P_\nu = \sum_{i \in M_\nu} \alpha_i .$$

Now applying inequality (3) we obtain at first

$$S_1 = \sum_{i \in M_1} \alpha_i^2 / a^{2i} \geqslant \bar{P}_1^2 / Q_1 \geqslant P_1^2 / Q_1 .$$

But since in view of (2) $P_1 = -\alpha$, we obtain inequality

(4) $$S_1 \geqslant \alpha^2 / Q_1 .$$

Next devide the set M_2 into subsets $M_2(1), M_2(2), \ldots$ in a natural way and apply (4) to vertices v_i $(i \in M_1)$. We obtain:

$$S_2(\tau) = \sum_{i \in M_2(\tau)} \alpha_i^2 / a^{2i} \geqslant \alpha_\tau^2 / Q_2(\tau),$$

and therefrom:

$$S_2 = \sum_\tau S_2(\tau) \geqslant \sum_\tau \alpha_\tau^2 / Q_2(\tau).$$

Now (3) yields

(5) $$S_2 \geqslant \Big(\sum_\tau |\alpha_\tau|\Big)^2 \Big/ \Big(\sum_\tau Q_2(\tau)\Big) \geqslant \alpha^2 / Q_2 .$$

Next by applying inequality (5) to vertices v_i $(i \in M_2)$, we similarly conclude that $S_3 \geqslant \alpha^2 / Q_3$.

In a general case, we obtain:

(6) $$S_l = \sum_{i \in M_l} \alpha_i^2 / a^{2i} \geqslant \alpha^2 / Q \,. \qquad (l = 1, 2, \ldots).$$

Therefrom it follows:

$$S=\sum_{l=1}^{\infty} S_l=\sum_{i\in\bigcup_\nu M_\nu}^{\infty} \alpha_i^2/a^{2i}\geqslant \sum_{\nu=1}^{\infty}\alpha^2/Q_\nu,$$

thus

$$S\geqslant\alpha^2\sum_{\nu=1}^{\infty}1/Q_\nu. \tag{7}$$

But since for every $n=1,2,\ldots$ the following is valid:

$$\sum_{\nu=1}^{n}1/Q_\nu\geqslant n^2/\sum_{\nu=1}^{n}Q_\nu>n^2\left(\sum_{\nu=1}^{\infty}Q_\nu\right)^{-1}=(1-a^2)\,n^2/a^2,$$

assuming that $\alpha\neq 0$ we get a contradiction from (7).

Hence $\alpha=\alpha_{j_0}=0$, for every edge $f_j\in E(G)$, and thus $x=0$.

Consequently, graph G cannot have $(D;\ -2)$ property.

2. Now assume that G has $(D;\ -2)$ property, there is exactly one (odd) cycle $i_1, i_2, \ldots, i_{2p+1}$ in G, and $x\neq 0$ is any D-eigenvector of B corresponding to the eigenvalue $\lambda=-2$.

Thus graph G consists of this cycle and of $2p+1$ subtrees $T_1, \ldots, T_{2p+1}$, some of which may be degenerate.

But then using the previous part of the statement for trees, one can easily infer that for every $f_j\in V(T_m)$ $(m=1,\ldots,2p+1)$ the corresponding coordinate $x_j=0$, and $\alpha_j+\alpha_{j+1}=0$ $(j=i_1,\ldots,i_{2p+1};\ i_{2p+2}=i_1)$. This obviously implies $\alpha_1=$ $=\ldots=\alpha_{2p+1}=0$, thus $x=0$, which is a contradiction. □

3. D-spectrum of line graphs of complete multipartite infinite graphs

Consider now any complete k-partite infinite graph $K=K(N_1,\ldots,N_q)$ $(q\geqslant 1)$ or complete infinite-partite infinite graph $K(N_1, N_2, \ldots)$. We briefly call the subsets N_l "parts" of K and index the vertices of each part N_l by numbers p_l $(p_l=1,2,\ldots,|N_l|\leqslant+\infty)$.

Next, label the edges of K by non-negative integers $1,2,\ldots$ in an arbitrary way. Let $j(m,\,l;\ p_l,\ p_m)=j(m,\,l;\,p_m,\,p_l)$ $(1\leqslant p_i\leqslant|N_i|\leqslant+\infty;\ i=1,\ldots,q;\ l\neq m)$ be the label of the edge joining vertex $p_l\in N_l$ with $p_m\in N_m$.

If then $x(l,\,m;\,p_l,\,p_m)$ are the coordinates of an arbitrary vector $x\in H$ in an orthonormal basis $\{e(l,\,m;\,p_l,\,p_m)\}$ of space H, we denote for brevity:

$$a^{j(l,\,m;\,p_l,\,p_m)}\,x(l,\,m;\ p_l,\ p_m)=y(l,\,m;\ p_l,\ p_m),$$

$$X_{l,\,m}(p_l)=\sum_{p_m=1}^{|N_m|}y(l,\,m;\ p_l,\ p_m),$$

and $X_{l,\,m}=\sum_{p_l}X_{l,\,m}(p_l)=X_{m,\,l}$ $(l\neq m)$.

Assuming next that the subset N_l is infinite, we have that the series $\sum_{p_l} X_{l,m}(p_l)$ is obviously absolutely convergent, so that $X_{l,m}(p_l)\to 0$ if $p_l\to 0$.

In the sequel, we shall use this fact several times.

We shall also often use the fact that any infinite sequence of natural numbers must converge to infinity. So, if for instance the subset N_m is infinite, we shall always have that $i(l, m;\ p_l, p_m)\to\infty$ as $p_m\to\infty$, and consequently $y(l, m;\ p_l, p_m)\to 0$.

Theorem 3. *Let G be the line graph $L(K_\infty)$ of the complete infinite graph K_∞. Then in any labelling of its vertex set,*

$$\sigma_D(G)=\{-2\}.$$

Proof. Of course, $-2\in\sigma_D(G)$ by Theorem 2.

Consider now an arbitrary indexing of the vertex set $V(L(K_\infty))=N$ by indices $i(p, q)=i(q, p)$ $(p, q\in N;\ p\neq q)$.

Let $\lambda\neq -2$ be any D-eigenvalue of graph G and $x\neq 0$ be a corresponding eigenvector. If $x(p, q)$ are the coordinates of x in a basis $\{e(p, q)\}$ $(p<q)$ of H, we have that the relation $Ax=\lambda Dx$ is equivalent with relation

$$(8)\qquad \sum_{r\neq p} a(p, r)\,x(p, r)+\sum_{r\neq q} a(q, r)\,x(q, r)=\alpha\, a(p, q)\,x(p, q)\quad (p\neq q)$$

where $a(p, q)=a^{i(p,q)}$ $(p\neq q)$, and $\alpha=\lambda+2$.

Now write (8) in the form

$$(9)\qquad X_p+X_q=\alpha\, a(p, q)\,x(p, q)\qquad (p\neq q),$$

where $X_p=\sum_{r\neq p} a(p, r)\,x(p, r)$, and let $q\to\infty$. Then $X_q\to 0$, $i(p, q)\to\infty$ and $a(p, q)\,x(p, q)\to 0$, so that necessarily $X_p=0$, for every $p\in N$.

Hence, for $\alpha\neq 0$ relation (9) yields $x(p, q)=0$ for every $p\neq q$, thus $x=0$, which is a contradiction. □

Theorem 4. *Let G be the line graph of complete bipartite graph $K(N_1, N_2)$. Then in any labelling of its vertex set:*

(a) $\sigma_D(G)=\{-2\}$, *if both N_1, N_2 are infinite;*

(b) $\sigma_D(G)=\{-2, a_1-2\}$, *if N_1 is finite and $|N_1|=a_1>1$;*

(c) $\sigma_D(G)=\{-1\}$, *if N_1 is finite and $|N_1|=1$.*

Proof. In view of Theorem 2, we obviously have that $-2\in\sigma_D(G)$ in the first two cases (a) and (b).

Consider an arbitrary indexing $i(1, 2; p, q) = i(2, 1; q, p)$ of the vertex set $V(G) = N$ ($p \leqslant |N_1| \leqslant +\infty$, $q \in N$) and a corresponding orthonormal basis $\{e(1,2; p, q)\}$ of H.

Case (a).-If

$$X_{1,2}(p) = \sum_{q=1}^{\infty} y(1, 2; p, q), \quad X_{1,2}(q) = \sum_{p=1}^{\infty} y(1, 2; p, q),$$

where $y(1,2; p, q) = a^{i(1,2;\, p,\, q)}\, x(1,2; p,q)$, then relation $Ax = \lambda\, Dx$ is equivalent to relation

$$X_{1,2}(p) + X_{1,2}(q) = \alpha\, y(1, 2; p, q) \qquad (p, q \in N). \tag{10}$$

Assuming that $\lambda \neq -2$ (i.e. $\alpha \neq 0$) and $x \neq 0$, let $q \to \infty$. Then $i(1, 2; p,q) \to \infty$ and $X_{1,2}(q) \to 0$, so that (10) gives $X_{1,2}(p) = 0$. Similarly $X_{1,2}(q) = 0$, which implies $x = 0$ (a contradiction).

Cases (b), (c).-We similarly find that relation $Ax = \lambda\, Dx$ ($\alpha = \lambda + 2 \neq 0$, $x \neq 0$) is equivalent to

$$X_{1,2}(p) + X_{1,2}(q) = \alpha\, y(1, 2; p, q) \qquad (p \leqslant |N_1|,\ q \in N). \tag{11}$$

Letting $q \to \infty$ we have that $i(1,2; p,q) \to \infty$, so that from (11) $X_{1,2}(p) = 0$. This yields

$$x_{1,2}(q) = \alpha\, y(1, 2; p, q) \qquad (q \in N). \tag{12}$$

Hence if $|N_1| = 1$ ($p = 1$), we obtain that $X_{1,2}(q) = \alpha\, X_{1,2}(q)$ and consequently $\alpha = 1$ (because $x(1,2; p,q) \neq 0$ for at least one $q \in N$). Thus $\lambda = -1$.

Moreover, it is easy to see that any vector $x \neq 0$ satisfying relation

$$\sum_{q=1}^{\infty} a^{i(1,2;\, 1,\, q)}\, x(1, 2; 1, q) = 0,$$

is a D-eigenvector corresponding to this D-eigenvalue.

Let now $|N_1| > 1$. Then since $x(1,2; p, q) \neq 0$ for some p, q, assuming that $\alpha \neq 0$ we find from (12) that $X_{1,2}(q) \neq 0$ (for *a* $q \in N$). But then immediately $\alpha = |N_1| = a_1$, thus $\lambda = a_1 - 2$.

Finally, construct at least one D-eigenvector corresponding to this D-eigenvalue. Take: $x(1,2; p, q) = 0$ ($q > 2$), $x(1,2; p,1) = a^{-i(1,2;\, p,\, 1)}$, $x(1, 2; p, 2) = -a^{-i(1,2,\, p,\, 2)}$ ($p = 1, 2, \ldots, a_1$). Then (11) is obviously satisfied with $\alpha = \lambda + 2 = a_1$, and since x is obviously finite, we get the statement. □

Theorem 5. *Let G be the line graph of complete k-partite infinite graph ($k \geqslant 2$), whose all parts are infinite. Then in any labelling of its vertex set*

$$\sigma_D(G)=\{-2\}.$$

Proof. We have that $G=L(K)$, where $K=K(N_1, \ldots, N_k)$ and all N_i are infinite.

Index the vertex set $V(L(K))=N$ by an arbitrary indexing $i(l, m;\ p_l, p_m)= =i(m, l;\ p_m, p_l)$ $(l, m=1, \ldots, k;\ l \neq m;\ p_l, p_m \in N)$ and choose an orthonormal basis $\{e(l, m; p_l, p_m)\}$ of H.

Then relation $Ax=\lambda Dx$ is equivalent to the following relation:

$$(13) \qquad \sum_{n \neq r} X_{r, n}(p_r) + \sum_{n \neq t} X_{t, n}(p_t) = \alpha x(r, t;\ p_r, p_t) \quad (r \neq t).$$

Putting in the above relation $p_t \to \infty$ we have that $i(r, t;\ p_r, p_t) \to \infty$, so that the first summand in (13) must be zero. Similarly, the second summand in this relation must be zero too, whence immediately $\alpha=0$, i.e. $\lambda=-2$.

Since obviously $-2 \in \sigma_D(G)$, the theorem is proved.□

In a quite similar way, we get the following.

Theorem 6. *If G is the line graph of a complete infinite-partite graph whose all components are infinite, then in any labelling of its vertex set, $\sigma_D(G)= =\{-2\}$.*

Theorem 7. *Let G be the line graph of a complete k-partite infinite graph ($k \geqslant 3$) with exactly one finite component. Then in any labelling of its vertex set, $\sigma_D(G)=\{-2\}$.*

Proof. Let $G=L(K)$, where $K=K(N_1, N_2, \ldots, N_k)$ and only N_1 is finite. Then in view of the presence of N_2, N_3, obviously $-2 \in \sigma_D(G)$.

Now choose an arbitrary indexing of the vertex set $V(G)=N$ by indices $i(r, t;\ p_r, p_t)$ $(r, t=1, \ldots, k;\ r \neq t;\ p_1 \leqslant |N_1|,\ p_i \in N,\ i \geqslant 2)$.

Then relation $Ax=\lambda Dx$ $(\lambda \neq -2, x \neq 0)$ is equivalent to the following two kinds of relations:

$$(14) \qquad \sum_{s=2}^{k} X_{1, s}(p_1) + \sum_{s \neq r} X_{r, s}(p_r) = \alpha x(1, r;\ p_1, p_r) \quad (r \geqslant 2),$$

$$(15) \qquad \sum_{s \neq r} X_{r, s}(p_r) + \sum_{s \neq t} X_{t, s}(p_t) = \alpha x(r, t;\ p_r, p_t) \quad (r \neq t;\ r, t \geqslant 2).$$

Now letting $p_t \to \infty$ in (15), we immediately have that all $x(r, t;\ p_r, p_t)=0$ $(r \neq t;\ r, t \geqslant 2)$.

Hence $X_{r, s}(p_r)=0$ $(2 \leqslant r, s \leqslant k)$, and since $\sum_{s \neq r} X_{r, s}(p_r)=0$ we obtain that $X_{r, 1}(p_r)=0$ $(r \geqslant 2)$.

Therefrom (14) becomes:

$$\sum_{s\neq 1} X_{1,s}(p_1) = \alpha\, x(1, r; p_1, p_r). \tag{16}$$

Letting now $p_r \to \infty$, we find from (16):

$$\sum_{s\neq 1} X_{1,s}(p_1) = \alpha\, x(1, r; p_1, p_r) = 0,$$

thus since $\alpha \neq 0$, $x(1, r; p_1, p_r) = 0$ $(r \geq 2)$ and consequently $x = 0$. □

Similarly, we have the following.

Theorem 8. *Let G be the line graph of a complete infinite-partite in-inite graph with exactly one finite component. Then in any labelling of its vertex et,* $\sigma_D(G) = \{-2\}$.

Theorem 9. *Let G be the line graph of a complete k-partite infinite graph $(k \geq 3)$ with exactly one infinite part. If these parts are $N_1, N_2, \ldots, N_k$ (N_1 — infinite), then in any labelling of its vertex set,*

$$\sigma_D(G) = \{-2, a_2 + \ldots + a_k - 2\},$$

where $a_i = |N_i|$ $(i = 2, \ldots, k)$.

Prof. Obviously $-2 \in \sigma_D(G)$ due to the presence of sets N_1, N_2, N_3.

Next, as in previous theorems, we get that relation $Ax = \lambda Dx$ is equivalent to the following two kinds of relations:

$$\sum_{s\neq l} X_{l,s}(p_l) + \sum_{s\neq m} X_{m,s}(p_m) = \alpha\, x(l, m; p_l, p_m) \qquad (2 \leq l, m \leq k;\ l \neq m), \tag{17}$$

$$\sum_{s\neq l} X_{l,s}(p_l) + \sum_{s\neq 1} X_{1,s}(p_1) = \alpha\, x(l, 1; p_l, p_1) \qquad (l \geq 2). \tag{18}$$

Letting $p_1 \to \infty$ in (18), we find

$$\sum_{s\neq l} x_{l,s}(p_l) = 0 \qquad (l = 2, \ldots, k),$$

so that (17) implies $x(l, m; p_l, p_m) = 0$ $(l, m \geq 2; l \neq m)$, thus $X_{l,m}(p_l) = 0$.

Consequently, (18) becomes

$$\sum_{s\neq 1} X_{1,s}(p_1) = \alpha\, x(l, 1; p_l, p_1) \quad (l \geq 2). \tag{19}$$

By summing the last relation in $p_l=1, \ldots, a_l=|N_l|$, we find

$$\sum_{s\neq 1} X_{1,s}(p_1)=\frac{\alpha}{a_l} X_{1,l}(p_1) \qquad (l=2, \ldots, k). \tag{20}$$

But since $X_{1,l}(p_1)\neq 0$ for at least one $l=2, \ldots, k$ and $p_1\in N$, because in the opposite case we would have from (18) (for $\alpha\neq 0$), $x(l, 1; p_l, p_1)=0$, thus $x=0$, the linear system (20) has a non-trivial solution for at least one p_l.

Hence

$$\det(p_{mr})=\det\left(1-\delta_r^m \frac{\alpha}{a_r}\right)=0 \qquad (m, r=2, \ldots k). \tag{21}$$

But $\alpha=0$ is a root of (21) whose multiplicity is $k-2$, and $\alpha=a_2+\ldots+a_k$ is a simple root of (21).

Assuming that $\alpha\neq 0$, we necessarly have that $\alpha=a_2+\ldots+a_k$.

We yet construct at least one D-eigenvalue corresponding to this D-eigenvalue.

Put: $x(l, 1; p_l, p_r)=0$ (if $p_l\geqslant 3$ or $p_r\geqslant 3$; $r=2, \ldots, k$), and

$$x(l, 1; p_l, p_r)=\frac{a_r}{a_2} a^{i(1,2;p_l,1)-i(1,r;p_l,p_r)} x(1, 2; p_l, 1)=$$

$$=\frac{a_r}{a_2}(-1)^{p_l-1} a^{i(1,2;p_l,1)-i(1,r;p_l,p_r)+i(1,2;3-p_r,1)}$$

$$(p_l, p_r=1, 2),$$

and all other coordinates are zero.

Then one can easily check that relations (17), (18) are valid with $\alpha=a_2+\ldots a_k$, and the corresponding eigenvector x is finite, thus the corresponding series for $\|x\|^2$ converges. This completes the proof. □

T h e o r e m 10. *Let G be the line graph of a complete infinite-partite graph with exact y one infinite part. Then in any labelling of its vertex set,* $\sigma_D(G)=\{-2\}$.

T h e o r e m 11. *Let G be the line graph of a complete k-partite infinite graph ($k\geqslant 4$) with at least two finite and at least two infinite parts. Then in any labelling of its vertex set,* $\sigma_D(G)=\{-2\}$.

P r o o f. Let $G=L(K)$ where $K=K(N_1, \ldots, N_p, N_{p+1}, \ldots, N_k)$ $(p\geqslant 2, k-p\geqslant 2)$, where $N_1, \ldots, N_p$ are finite and $N_{p+1}, \ldots, N_k$ are infinite. Then due to the presence of N_{p+1}, N_{p+2}, obviously $-2\in\sigma_D(G)$.

Next, relation $Ax=\lambda Dx$ is equivalent to the following three sets of relations:

$$(22)\qquad \sum_{s\neq l} X_{l,s}(p_l)+\sum_{s\neq m} X_{m,s}(p_m)=\alpha x(l, m;\ p_l, p_m)\qquad (l, m=1, \ldots, p;\ l\neq m)$$

$$(23)\qquad \sum_{s\neq l} X_{l,s}(p_l)+\sum_{s\neq r} X_{r,s}(p_r)=\alpha x(l, r;\ p_l, p_r)\qquad (l=1, \ldots, p;\ r\geqslant p+1)$$

$$(24)\qquad \sum_{s\neq r} X_{r,s}(p_r)+\sum_{s\neq t} X_{t,s}(p_t)=\alpha x(r, t;\ p_r, p_t)\qquad (r, t\geqslant p+1;\ r\neq t).$$

From (24), we easily find that for $\alpha\neq 0$ it follows

$$(25)\qquad \sum_{s\neq r} X_{r,s}(p_r)=0,$$

and consequently $x(r, t;\ p_r, p_t)=0\quad (r, t\geqslant p+1;\ r\neq t)$.

Substituting now (25) in (23) and letting $p_r\to\infty$ we find

$$(26)\qquad \sum_{s\neq l} X_{l,s}(p_l)=0,$$

so that $\alpha\neq 0$ implies $x(l, r;\ p_l, p_r)=0\quad (l\leqslant p;\ r\geqslant p+1)$.

Finally, substituting (26) in (22), we get $x(l, m;\ p_l,\ p_m)=0\quad (l, m\leqslant p;\ l\neq m)$, and hence $x=0$ (contradiction).

This complets the proof.□

T h e o r e m 12. *Let G be the line graph of a complete infinite-partite graph with at least two infinite and at least two finite parts. Then in any labelling of its vertex set,* $\sigma_D(G)=\{-2\}$.

We end with a problem.

P r o b l e m. *Determine the D-spectrum of the line graph $G\neq L(K_\infty)$ of a complete infinite-partite graph whose all parts are finite.*

So, we calculated the D-spectrum of line graphs of all complete k-partite and infinite-partite infinite graphs, except infinite-partite graphs whose all parts are finite. Namely, the above methods were not suitable to be applied to this remaining case.

We also note that, although the applied methods have some similarities, we were not able to give a unique proof for all cases considered.

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