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# EQUATION OF HEAT CONDUCTION WITH FINITE WAVE SPEEDS

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## 1. Introduction

In the introduction of his paper "A General Theory of Heat Conduction with Finite Wave Speeds, M.E. Gurtin and A.C. Pipkin [1] wrote: " The classical linear theory of heat conduction for homogeneous and isotropic media is based on, the equation: $a\Delta U = \dot{U}$, where $U = \dot{U}(x, t)$ is the absolute temperature, $\dot{U} = \dfrac{\partial U}{\partial t}$, $\Delta$ is the Laplacian and $a>0$ is a constant. This equation, which is parabolic, has a very unpleasant feature: a thermal disturbance at any point in the body is, felt instantly at every other point; or in terms more suggestive than precise, the speed of propagation of disturbances is infinite.

In this paper we develop a general theory of heat conduction for nonlinear materials with memory, a theory which has associated with it finite propagation speeds."

Let $a(t)$ be the heat-flux relaxation function, $b(t)$ the energy relaxation function, the number $c$ the instantaneous specific heat at wave, $r(x,t)$ the heat supplied to the body by the external world and $U(x,t)$ the absolute temperature, then the mentioned authors propose the following integro-differential equation:

$$\text{(1)} \qquad c\,\ddot{U}(x, t) + b(0)\,\dot{U}(x. t) + \int_0^\infty b'(s)\dot{U}(x, t-s)\,ds =$$

$$= a(0)\,\Delta U(x, t) + \int_0^\infty a'(s)\,\Delta U(x, t-s)\,ds + \dot{r}(x, t)$$

instead of the classical one: $a\,\Delta U(x, t) = \dot{U}(x, t)$.

We shall analyse equation (1) in case of one dimension and apply our results [4], [5] to it.

## 2. Translation of equation (1) in the field of Mikusiński operators

We shall really deal with the equation:

$$(1') \qquad c\,U_t''(x,t)+b(0)\,U_t'(x,t)+\int_0^t b'(s)\,U_t'(x,t-s)\,ds=$$

$$=a(0)\,U_x''(x,t)+\int_0^t a'(s)\,U_x''(x,t-s)\,ds+r_t'(x,t)$$

$U(x,t)=0$, $t<0$, $x$ belongs to an interval J (J can be an unbounded interval too). We can suppose that $U(x,t)=\text{const.}$ for $t<0$, and this is really the same problem.

Before we start to treat equation (1′) we shall give:

**Some notations and remarks:** $\mathscr{L}$ is the ring of local integrable functions defined over $[0,\infty)$ with sum and composition $(\mathbf{f}\,\mathbf{g}=\{\int_0^t f(t-u)\,g(u)\,du\})$; $\mathbf{f}=\{f(t)\}$ is an element from $\mathscr{L}$ which corresponds to the numerical function $f(t)$. $\mathscr{M}$ is the field of Mikusiński operators which enlarge the ring $\mathscr{L}$. $\mathscr{L}(x)$ is the set of all mappings $\{f(x,t)\}: I\to\mathscr{L}$, $I=[0,x_1]$, where $f(x,t)$ is bounded and measurable over $I$ for $t>0$. $J(x)$ is an algebra of those elements which have the following form: $\mathbf{a}(x)=\mathbf{q}\{f(x,t)\}$, $\mathbf{q}\in\mathscr{M}$, $\{f(x,t)\}\in\mathscr{L}$ for every $x\in I$. For the definition of the derivative and integral in $\mathscr{M}(x)$ see [2].

Let us remark that J. Wloka [8] constructed "operator-distributions" which behave in the variable $t$ as a Mikusiński operator and in $x$ as a distribution of the finite order. He proved the following theorem: "Let $x(\lambda)$ be a solution of the equation $\sum_{i=0}^{m} a_i D^i x(\lambda)=f(\lambda)$, $a_i\in\mathscr{M}$, in the space of operator-distributions ($D^i x(\lambda)$ is the derivative in this space), then $x(\lambda)=D^n\mathbf{X}(\lambda)$ where $\mathbf{X}(\lambda)$ is a solution of the equation $\sum_{i=0}^{m}\mathbf{a}_i\mathbf{X}^{(i)}(\lambda)=\mathbf{F}(\lambda)$ in $\mathscr{M}$ where $f(\lambda)=D^n\mathbf{F}(\lambda)$". A consequence of Wloka's result is that in our equation or in initial and boundary conditions we can have distributions as $\delta$, $\delta^{(k)}$, ... which appear very frequently in mathematical models of physics.

Now we can deal with our equation (1′). To this equation corresponds in the field $\mathscr{M}$ of Mikusiński operators the following equation:

$$(2) \qquad \mathbf{a}\,\mathbf{U}''(x)-(\mathbf{c}+\mathbf{b})\,\mathbf{s}\,\mathbf{U}(x)=-c\,U(x,0)\,\mathbf{I}-\mathbf{b}\,U(x,0)-lc\,U_t'(x,0)$$

$$-\mathbf{r}(x)+lr(x,0)\equiv\mathbf{f}(x)\qquad x\in J.$$

We used here the known formula in $\mathscr{M}$:

$$\{f^{(n)}(t)\}=\mathbf{s}^n\,\mathbf{f}-\mathbf{s}^{n-1}f(0)-\cdots-f^{(n-1)}(0)\,\mathbf{I}$$

and the notations in $\mathcal{M}$: $\mathbf{s}$ is the differential operator, $l$ the integral operator and $\mathbf{I}$ is the unit element in $\mathcal{M}$.

## 3. General solution of equation (2)

We suppose: a) $c>0$ and $a(0)>0$; b) The operators $\mathbf{a}$ and $\mathbf{b}$ are algebraic operators, that means they are given by the series:

(3)
$$\mathbf{a}=\sum_{k=p}^{\infty}\alpha_k\, l^{k/p},\qquad \alpha_p>0$$
$$\mathbf{b}=\sum_{k=q}^{\infty}\beta_k\, l^{k/p},\quad q\geqslant p,\quad \beta_q\neq 0$$

convergent in $\mathcal{M}$. The corresponding functions $a(t)$ and $b(t)$ to the operators $\mathbf{a}$ and $\mathbf{b}$ are:

(3′)
$$a(t)=\sum_{k=p}^{\infty}\alpha_k\frac{t^{k/p-1}}{\Gamma(k/p)},\qquad a(0)=\alpha_p>0,$$
$$b(t)=\sum_{k=q}^{\infty}\beta_k\frac{t^{k/p-1}}{\Gamma(k/p)},\qquad q\geqslant p,\quad \beta_q\neq 0;$$

these series converge for all $t$.

The homogeneous part of equation (2) is:

(4)
$$\mathbf{a}\,\mathbf{u}''(x)-(c+\mathbf{b})\,\mathbf{s}\,\mathbf{u}(x)=0$$

The correspond characteristic equation $\mathbf{a}\,\mathbf{r}^2-(c+\mathbf{b})\,\mathbf{s}=0$ gives:

(5)
$$\mathbf{r}_{1,2}=\pm\sqrt{\frac{c+\mathbf{b}}{\mathbf{a}}}\sqrt{\mathbf{s}}=\pm\sqrt{\frac{c+\mathbf{b}}{\hat{\mathbf{a}}}}\,\mathbf{s}\equiv\pm\mathbf{r}\quad ,$$

where $\hat{\mathbf{a}}=\mathbf{s}\,\mathbf{a}=\sum_{k=0}^{\infty}\alpha_{p+k}\, l^{k/p}$. $\mathbf{r}$ is an algebraic operator too, given by a power series in $l^{1/p}$:

(6)
$$\mathbf{r}=\sqrt{\frac{c+\mathbf{b}}{\hat{\mathbf{a}}}}\,\mathbf{s}=\mathbf{s}\sum_{k=0}^{\infty}\gamma_k\, l^{k/p}$$

because the field of algebraic operators is algebraicaly closed [2]. We also know that $\mathbf{r}$ is a logarithm [2].

Now the two linearly independent solutions of equation (4) are: $\exp(x\mathbf{r})$ and $\exp(-x\mathbf{r})$. The general solution of equation (4) is:

$$\mathbf{U}(x)=\mathbf{c}_1 \exp(-x\,\mathbf{r})+\mathbf{c}_2 \exp(x\,\mathbf{r}),\ \mathbf{c}_1, \mathbf{c}_2 \in \mathcal{M}. \tag{7}$$

Remark: If we suppose that $a(0)$ can equal zero, then the operator $\mathbf{r}$ has the form $\mathbf{r}=\sqrt{\frac{c+\mathbf{b}}{\hat{\mathbf{a}}}}\,\mathbf{s}^{\nu}$, $\nu>1$. In this case $\mathbf{r}$ is not a logarithm [2], and there is no solution in $\mathcal{M}$ of equation (4). The supposition $a(0)\neq 0$ has as a consequence another fact too. Namely, from all the possibilities that $\mathbf{a}$ has an expression of the form $\mathbf{a}=\sum_{k=0}^{\infty} \alpha_k l^{k\omega}$, $\omega\in R^+$, we have to choose only one: $\omega=l/p$, $p\in N$, $\alpha_0=\cdots=\alpha_{p-1}=0$. $\alpha_p\neq 0$.

For the construction of the general solution (7) it remains only to find the numbers $\gamma_k$, the coefficients in series (6). For this purpose we can use the same procedure as for the power series, and the relation from which we can find our $\gamma_k$ is:

$$\sum_{m=0}^{k} \alpha_{p+k-m} \sum_{j=0}^{m} \gamma_j \gamma_{m-j}=\beta_k, \qquad k=0, 1, \ldots, \quad \beta_0=c \tag{8}$$

In such a way:

$$\gamma_0=\sqrt{\frac{c}{\alpha_p}}, \qquad \gamma_1=\frac{1}{2}\sqrt{\frac{c}{\alpha_p}}\left(\frac{\beta_1}{c}-\frac{\alpha_{p+1}}{\alpha_p}\right), \ldots \tag{8'}$$

Lemma 1. *If* $\frac{\beta_m}{c}-\frac{\alpha_{p+m}}{\alpha_p}=0$, $m=1, 2, \ldots, q-1$ *and* $\frac{\beta_q}{c}-\frac{\alpha_{p+q}}{\alpha_p}\neq 0$, *then* $\gamma_1=\cdots=\gamma_{q-1}=0$, $\gamma_q=\frac{1}{2}\sqrt{\frac{c}{\alpha_p}}\left(\frac{\beta_q}{c}-\frac{\alpha_{p+q}}{\alpha_p}\right)$.

Proof. First we prove the case $q=2$. From relation (8) follows that $\beta_2=\alpha_2\gamma_0^2+2\,\alpha_1\gamma_1\gamma_0+\alpha_0(\gamma_1^2+2\,\gamma_2\gamma_0)$; from (8′) we have $\gamma_1=0$ and

$$\gamma_2=\frac{1}{2}\sqrt{\frac{c}{\alpha_p}}\left(\frac{\beta_2}{c}-\frac{\alpha_{p+2}}{\alpha_p}\right).$$

Let us suppose that our Lemma is true for $m=k\geqslant 2$, we shall prove that it is true for $m=k+1$ too. In this case

$$\beta_{k+1}=\sum_{m=0}^{k+1} \alpha_{p+k+1-m} \sum_{j=0}^{m} \gamma_j \gamma_{m-j}=\alpha_{p+k+1}\gamma_0^{\,2}+2\,\gamma_0\gamma_{k+1}\alpha_p$$

and

$$\gamma_{k+1}=\frac{1}{2}\sqrt{\frac{c}{\alpha_p}}\left(\frac{\beta_{k+1}}{c}-\frac{\alpha_{p+k+1}}{\alpha_p}\right).$$

The two solutions $\exp(\pm x\mathbf{r})$ of equation (4) can be written in the form:

$$\exp(\pm x\mathbf{r})=\exp(\pm x\gamma_0\mathbf{s})\exp\left(\pm x\sum_{i=1}^{p-1}\gamma_i l^{i/p-1}\right)\exp(\pm x\gamma_p)(\mathbf{I}+\{F(x,t)\}), \tag{9}$$

where $\{F(x, t)\}\in\mathscr{L}(x)$. To analyse such a solution it is enough to know the meaning of every of the following products: $\exp(\pm x\gamma_0\mathbf{s})$; $\exp(\pm x\gamma_k\mathbf{s}^{1-k/p})$, $k=1,\ldots,p-1$; $\exp(\pm x\gamma_p)$; $\exp(\pm x\gamma_k l^{k/p-1})$, $k>p$.

We know that $\exp(-x\gamma_0\mathbf{s})$, for $x\gamma_0>0$, is the shift operator; for $x\gamma_k>0$, and $1>1-k/p>0$ is:

$$\exp(-x\gamma_k\mathbf{s}^{1-k/p})=\begin{Bmatrix} t^{-1}\Phi(0,-(1-k/p),-x\gamma_k t^{-(1-k/p)}), & t\neq 0\\ 0 & t\neq 0\end{Bmatrix} \tag{10}$$

and

$$\exp(\pm\gamma_k x l^{k/p-1})=\{\Phi(0,k/p-1,\pm x\gamma_k t^{k/p-2})\}-I,\ \frac{k}{p}-1>0 \tag{11}$$

where

$$\Phi(\beta,\rho,z)=\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(i+1)\Gamma(\rho i+\beta)} \tag{12}$$

is the function of E.M. Wright [9] which is defined for $\rho>0$ or $0>\rho>-1$ and $\beta$ a complex number.

Further we have to know that the inverse element of $f\in\mathscr{L}$ can not be in $\mathscr{L}$.

The following lemma, proved in [7] can be useful in finding the analytical expression of solution (7).

Lemma 2. *Let us suppose that* $c_1>0$, $0<\alpha<1$ *and* $1-k\alpha>0$, *then the operator:*

$$\exp(-|x|c_1\mathbf{s}^{1-\alpha}+\cdots+|x|c_k\mathbf{s}^{1-k\alpha})$$

*is defined by the continuous function:*

$$\frac{1}{2\pi i}\int_{x_2-i\infty}^{x_2+i\infty}\exp(tz)\exp\left(-|x|\sum_{i=1}^{k}c_i z^{1-i\alpha}\right)dz \tag{12'}$$

*for* $t\geqslant 0$ *and* $x\neq 0$.

Let us suppose now that we have two linearly independent solutions of the equation (4): $U_1(x)$ and $U_2(x)$, then a particular solution of the equation (2) is:

$$\mathbf{U}_0(x) = \frac{-1}{2\mathbf{r}} \mathbf{U}_1(x) \int_{x_0}^{x} \mathbf{U}_1^{-1}(y)\, \mathbf{f}(y)\, dy + \tag{13}$$

$$+ \frac{1}{2\mathbf{r}} \mathbf{U}_2(x) \int_{x_0}^{x} \mathbf{U}_2^{-1}(y)\, \mathbf{f}(y)\, dy$$

and the general solution of equation (2) is:

$$\mathbf{U}(x) = \mathbf{c}_1 e^{-\mathbf{r}x} + \mathbf{c}_2 e^{\mathbf{r}x} - \frac{1}{2\mathbf{r}} \int_{x_0}^{x} [e^{-(x-y)\mathbf{r}} - e^{(x-y)\mathbf{r}}]\, \mathbf{f}(y)\, dy. \tag{14}$$

## 4. Bounded bar

We shall suppose now that our equation (1') describes the temperature for a bounded bar. That means that the interval $J$ can be taken as $J = [0, x_1]$.

Theorem 1. *Let us suppose that*

*1. $c$ is a positive number; 2. the functions $a(t)$ and $b(t)$ have the form given by (3'); 3. the function $\{r_t'(x, t)\} \in \mathscr{L}$ for $x \in [0, x_1]$ and $\{r(x, t)\}$ belongs to $\mathscr{L}(x)$.*

*Then equation (1') has a unique generalized solution (16) which satisfies the conditions:*

(15) (15') $U(x, 0) = u_0(x)$, $U_t'(x, 0) = u_1(x)$ (15'') $U(0, t) = v_0(t)$, $U_x'(0, t) = v_1(t)$

*where $v_0, v_1 \in \mathscr{L}$ and $u_0(x)$, $u_1(x)$ are integrable functions over* $[0, x_1]$.

Proof. To equation (1') corresponds in $\mathscr{M}$ equation (2). The general solution of equation (2) is given by (14).

We know that our operator $\exp(\pm x\mathbf{r})$ exists for $r$ given by (6). In relation (9) we decomposed this operator in factors for which we discussed when such a factor is given by a numerical function from $\mathscr{L}$ or is the inverse of it.

We can appropriate our constant operators $\mathbf{c}_1$ and $\mathbf{c}_2$ from the general solution (14) in such a way that conditions (15) are satisfied. The solution of equation (2) which satisfies conditions (15) is:

$$\mathbf{U}(x)=\frac{1}{2\mathbf{r}}(v_0\mathbf{r}-\mathbf{v}_1)e^{-x\mathbf{r}}+\frac{1}{2\mathbf{r}}(\mathbf{v}_0\mathbf{r}+\mathbf{v}_1)e^{x\mathbf{r}}+ \tag{16}$$
$$+\frac{1}{2\mathbf{r}}e^{x\mathbf{r}}\int_0^x e^{-y\mathbf{r}}\mathbf{f}(y)\,dy-\frac{1}{2\mathbf{r}}\int_0^x e^{-(x-y)\mathbf{r}}\mathbf{f}(y)\,dy$$

where $\mathbf{f}(y)$ is given by relation (2).

Theorem 2. *Let us suppose that the conditions of Theorem* 1 *are satisfied and moreover: if the number $p$ from $a(t)$ is greater than one, then $\alpha_{p+m}$ has to vanish for all $m$, $1\leqslant m\leqslant p-1$, or for the least $m$, $1\leqslant m<p$, for which $\alpha_{m+p}$ differs from zero, these coefficients have to be negative.*

*Then solution (16) is given by a numerical function belonging to $\mathcal{L}(x)$ if and only if there exists $\{H(x,t)\}\in\mathcal{L}(x)$ such that relations (20) and (21) are satisfied.*

Before we prove our Theorem 2 we give four lemmas.

Lemma 3. *Let us suppose that $\{P(x,t)\}$ and $\{Q(x,t)\}$ belong to the set $\mathcal{L}$ for every $x\in[0,x_1]$. If $e^{xs}\{P(x,t)\}=\{Q(x,t)\}$, then $P(x,t)=0$, $0\leqslant t\leqslant x$.*

Proof. The relation between $P$ and $Q$ can be written: $\{P(x,t)\}=$ $=e^{-xs}\{Q(x,t)\}$ and the proof of the lemma follows from the definition of the shift operator $e^{-xs}$, $x>0$.

Lemma 4. *There exists no function $\{G(x,t)\}$ which belongs to $\mathcal{L}$ for every $x\in[0,x_1]$, such that $e^{\pm xs}\{P(x,t)\}^{-1}=\{G(x,t)\}$, where $\{P(x,t)\}\in\mathcal{L}$ for $x\in[0,x_1]$.*

Proof. From our relation between $P$ and $G$ follows $e^{\pm xs}=\{G(x,t)\}$ $\{P(x,t)\}$, but this is impossible because $e^{\pm xs}$ is not from the ring $\mathcal{L}$.

Lemma 5. *We suppose that $\{f(x,t)\}\in\mathcal{L}(x)$, $x\in[0,x_1]$; $f(x,t)=0$, $t<0$, $x\in[0,x_1]$, and $\gamma>0$, then*

$$\int_0^x e^{-\gamma ys}\{f(y,t)\}\,dy=\frac{1}{\gamma}\left\{\int_0^t \hat{f}\left(\frac{u}{\gamma},\ t-u\right)du\right\}$$

*where*

$$\hat{f}(z,t)=\begin{cases}f(z,t), & z\in[0,x]\\ 0 & z\notin[0,x]\end{cases}\quad t\in R\Bigg\}$$

Proof.

$$\int_0^x e^{-\gamma y s}\{f(y,t)\}\,dy=\frac{1}{\gamma}\int_0^{x\gamma} e^{-us}\left\{f\left(\frac{u}{\gamma},t\right)\right\}du$$

$$=\begin{Bmatrix}\frac{1}{\gamma}\int_0^t f\left(\frac{u}{\gamma},t-u\right)du, & 0\leqslant t\leqslant \gamma x\\ \frac{1}{\gamma}\int_0^{x\gamma} f\left(\frac{u}{\gamma},t-u\right)du, & t\geqslant \gamma x\end{Bmatrix}$$

$$=\frac{1}{\gamma}\left\{\int_0^t \hat{f}\left(\frac{u}{\gamma},t-u\right\}du.$$

Consequences of Lemma 5:

1. $$\int_0^x e^{-\gamma y s} g(y)\{h(t)\}\,dy=\frac{1}{\gamma}\left\{\hat{g}\left(\frac{t}{\gamma}\right)\right\}\{h(t)\};$$

2. $$\int_0^x e^{-\gamma y s} f(y)\,\mathbf{I}\,dy=s\int_0^x e^{-\gamma y s} f(y)\,\{1\}\,dy=\frac{1}{\gamma}\left\{\hat{f}\left(\frac{t}{\gamma}\right)\right\};$$

3. $$\int_0^x e^{-\gamma y s}\, g(y)\,\{e^{yf(t)}\}\,dy=\sum_{k=0}^{\infty}\frac{1}{k!}\left\{g\left(\frac{t}{\gamma}\right)\left(\frac{t}{\gamma}\right)^k\right\}\{f(t)\}^k\in\mathscr{L};$$

4. $$\int_{x_1}^{x_2} e^{-\gamma y s}\{f(y,t)\}\,dy=\frac{1}{\gamma}\left\{\int_0^t \underset{\hat{}}{\hat{f}}\left(\frac{u}{\gamma},t-u\right)du\right\},\ x_1,\ x_2\geqslant 0;$$

*where*

$$\underset{\hat{}}{\hat{f}}(z,t)=\begin{cases} f(z,t), & z\in[x_1,x_2]\\ 0, & z\notin[x_1,x_2]\end{cases}\quad t\in R.$$

5. $$\int_{x_1}^{x_2} exp[-(y-x_1)\gamma\,\mathrm{s}]\{f(y,t)\}\,dt=$$

$$=\frac{1}{\gamma}\left\{\int_0^t \underset{\hat{}}{\hat{f}}(x_1+\frac{\mathrm{u}}{\gamma},\ t-u)\,du\right\}.$$

Remark. Lemma 5 is a slight generalization of a proposition proved in [2]. This proposition says exactly the same thing as consequence 2 of our Lemma 5.

Analogously to Lemma 5 we have:

Lemma 6. *Let a new property of the function $f(x, t)$, namely $f(x, t) = 0$, $x<0$, $t \in R$ be satisfied besides the suppositions of Lemma 5, then:*

$$\int_0^x e^{-(x-y)\gamma s}\{f(y, t)\}\,dy = \frac{1}{\gamma}\left\{\int_0^t f\left(x-\frac{\tau}{\gamma}, t-\tau\right)d\tau\right\}.$$

Proof.

$$\int_0^x e^{-(x-y)\gamma s}\{f(y, t)\}\,dy = \frac{1}{\gamma}\int_0^{x\gamma} e^{-us}\left\{f\left(x-\frac{u}{\gamma}, t\right)\right\}du$$

$$= \frac{1}{\gamma}\int_0^{\gamma x}\left\{f\left(x-\frac{u}{\gamma}, t-u\right)\right\}du$$

$$= \frac{1}{\gamma}\left\{\int_0^t f\left(x-\frac{u}{\gamma}, t-u\right)du\right\}.$$

Consequences of Lemma 6.:

1. *If* $\{f(x, t)\} = g(y)\{h(t)\}$ *and* $g(y) = 0$, $y<0$, *then*

$$\int_0^x e^{-(x-y)\gamma s} g(y)\{h(t)\}\,dy = \frac{1}{\gamma}\left\{g\left(x-\frac{t}{\gamma}\right)\right\}\{h(t)\};$$

2. $$\int_0^x e^{-(x-y)s\gamma} F(y)\,\mathbf{I}\,dy = \frac{1}{\gamma}\left\{F\left(x-\frac{t}{\gamma}\right)\right\},\ F(y) = 0,\ y<0$$

3. $$\int_0^x e^{-(x-y)\gamma s} g(y)\{e^{yf(t)}\}\,dy \in \mathscr{L} \text{ for } x \geq 0 \text{ where } f(t) = g(t) = 0,\ t<0.$$

Proof of Theorem 2. To give necessary and sufficient conditions for solution (16) to belong to $\mathscr{L}(x)$, we have to know a little bit more about the operator $\frac{1}{\mathbf{r}}$ and operator functions as $e^{-x\mathbf{r}}$ as $f(y)$. We shall start with $\frac{1}{\mathbf{r}}$:

$$\frac{1}{\mathbf{r}} = \frac{1}{\mathbf{s}\sqrt{\frac{c+\mathbf{b}}{\hat{\mathbf{a}}}}} = l\sqrt{\frac{\hat{\mathbf{a}}}{c+\mathbf{b}}} = \sum_{i=0}^{\infty} d_i\, l^{1+i/p} \tag{17}$$

which shows that $\frac{1}{\mathbf{r}}$ belongs to the set $C \subset \mathscr{L}$ of operators defined by a continuous function for all $t \geqslant 0$.

In relation (9) we decomposed the operator $e^{-x\mathbf{r}}$. By our suppositions: $\gamma_0 > 0$ and $\exp(-\gamma_0 x\mathbf{s})$ is the shift operator;

$\exp\left(-x \sum_{i=1}^{p-1} \gamma_i \mathbf{s}^{1-i/p}\right)$ can be $\mathbf{I}$ or given by a continuous function (see supposition 3, Lemma 2 and relation (10)). In such a way $e^{-x\mathbf{r}}$ has the form:

$$e^{-x\mathbf{r}} = \exp(-\gamma_0 x \mathbf{s})(k_1 \mathbf{I} + k_2\{g(x, t)\}) \exp(\gamma_p x) \tag{18}$$

where $k_1$ and $k_2$ are complex numbers and the function $\{g(x, t)\}$ belongs to $\mathscr{L}(x)$.

The operator function $\mathbf{f}(y)$ (see relation (2)) has the form:

$$\mathbf{f}(y) = g_1(y)\mathbf{I} + \{g_2(y, t)\} \tag{19}$$

where $g_1(y)$ is integrable over $[0, x_1]$ and $\{g_2(y, t)\} \in \mathscr{L}(x)$.

Taking care of the forms of $\frac{1}{\mathbf{r}}$, $e^{-x\mathbf{r}}$ and $\mathbf{f}(y)$ we see that the first addend in our solution (16) is an element from $\mathscr{L}(x)$. By Lemma 6 the last addend in relation (16) is an element from $\mathscr{L}(x)$ too. Consequently, solution (16) belongs to $\mathscr{L}(x)$ if and only if:

$$\frac{1}{2\mathbf{r}}\left(\mathbf{v}_0 \mathbf{r} + \mathbf{v}_1 + \int_0^x e^{-y\mathbf{r}} \mathbf{f}(y)\, dy\right) e^{x\mathbf{r}} \in \mathscr{L}(x);$$

this is equivalent to the statement that there exists $\{H(x, t)\} \in \mathscr{L}(x)$ such that:

$$\frac{1}{2\mathbf{r}}\left[\mathbf{v}_0 \mathbf{r} + \mathbf{v}_1 + \int_0^x e^{-y\mathbf{r}} \mathbf{f}(y)\, dy\right] \exp(\gamma_0 x \mathbf{s}) = \tag{20}$$

$$= \exp\left(-x \sum_{i=1}^{p-1} \gamma_i \mathbf{s}^{1-i/p}\right) \{H(x, t)\}.$$

Finally, by Lemma 3, it follows that:

$$\frac{1}{2\mathbf{r}}\left(\mathbf{v}_0 \mathbf{r} + \mathbf{v}_1 + \int_0^x e^{-y\mathbf{r}} \mathbf{f}(y)\, dy\right) = 0, \quad 0 \leqslant t \leqslant \gamma_0 x. \tag{21}$$

Remarks. 1. In reality relations (20) and (21) say that the conditions (15) can not be chosen in an arbitrary way if we wish to have our solution $\mathbf{U}(x)$ defined by a numerical function.

2. Solution (16), being from $\mathcal{L}(x)$ need not be a classical one because our derivatives are taken in a generalized sense.

3. In case $p=1$ the situation is much more simple. The suppositions of Theorem 2 are the same as those from Theorem 1, and instead of conditions given by relations (20) and (21) we have only what is given by the second one.

The root $\mathbf{r}$ of the characteristic equation is now:

$$r=\gamma_0 \mathbf{s}+\gamma_1+\sum_{i=1}^{\infty} \gamma_{i+1} l^i$$

and the linearly independent solutions:

$$\exp(\pm x\mathbf{r})=\exp(\pm\gamma_1 x)\exp(\pm\gamma_0 x\mathbf{s})\exp(\pm x\{g(t)\})$$

where $g(t)$ is a function which has all derivatives for $t\geqslant 0$.

Relation (20) becomes now

$$\frac{1}{2\mathbf{r}}\left(\mathbf{v}_0\mathbf{r}+\mathbf{v}_1+\int_0^x e^{-y\mathbf{r}}\mathbf{f}(y)\,dy\right)\exp(\gamma_0 x\mathbf{s})=\{H(x,t)\}.$$

And we do not demand a special form of the right side of relation (20) as in the case $p>1$. It remains only to require that relation (21) be true. In such a way, in case $p=1$, relation (21) gives a sufficient and necessary condition for the solution $\mathbf{U}(x)$ to be a numerical function.

## 5. Boundary value problem

We shall use our results published in [6] to solve the boundary value problem for equation (1′).

Let $L[\mathbf{u}(x)]$ be the differential expression:

$$L[\mathbf{u}(x)]=[\mathbf{a}(x)\mathbf{u}'(x)]'-\mathbf{b}(x)\mathbf{u}(x),\quad \mathbf{a}(x)\neq 0,\ x\in[0.\ x_1]$$

where $\mathbf{a}(x)$ and $\mathbf{b}(x)$ have a derivative in $J(x)$, then we have the following proposition [6]:

„If the equation $L[\mathbf{u}(x)]=0$ has two solutions $\mathbf{u}_1(x)$ and $\mathbf{u}_2(x)$ in $J(x)$, one of which satisfies boundary condition (22′)

(22) (22′) $\mathbf{u}(0)=0$ (22″) $\mathbf{u}(x_1)=0$

and the other (22″),

$$\mathbf{u}_1(x)\,\mathbf{u}_2'(x)-\mathbf{u}_1'(x)\,\mathbf{u}_2(x)\neq 0$$

for at least one $x \in (0, x_1)$, then $L[\mathbf{u}(x)] = -\mathbf{f}(x)$, $\mathbf{f}(x) \in \mathscr{L}(x)$, has a unique solution:

$$\mathbf{u}(x) = \int_0^{x_1} \mathbf{G}(x, y)\, \mathbf{f}(y)\, dy, \quad x \in (0, x_1)$$

under boundary condition (22), where $\mathbf{G}(x, y)$ is „Green's function"."

First we shall construct „Green's function" corresponding to the differential expression $L[\mathbf{U}(x)] = \mathbf{a}\, \mathbf{U}''(x) - (c+\mathbf{b})\, \mathbf{s}\, \mathbf{U}(x)$ and boundary conditions (22).

The particular solutions which satisfy conditions (22′) and (22″) respectively are:

$$\mathbf{U}_1(x) = \exp(\mathbf{r}\, x) - \exp(-\mathbf{r}\, x)$$

$$\mathbf{U}_2(x) = \exp[-(x_1 - x)\, \mathbf{r}] - \exp[(x_1 - x)\, \mathbf{r}].$$

We know that „Green's function" exists if we have $\exp(-2\, x_1\, \mathbf{r}) \neq \mathbf{I}$. In [3] it is proved that $\exp(-2\, x_1\, \mathbf{r}) = 1$ if and only if $2\, x_1\, \mathbf{r} = 2\, k\, \pi\, i$. Our $\mathbf{r}$ always differs from $k\, \pi\, i / x_1$. The „Green's function" for $L[\mathbf{U}(x)]$ is:

$$\text{(23)} \qquad \mathbf{G}(x, \xi) = \begin{cases} -\dfrac{1}{\mathbf{c}}\, \mathbf{U}_2(\xi)\, \mathbf{U}_1(x), & x \leqslant \xi \\[2ex] -\dfrac{1}{\mathbf{c}}\, \mathbf{U}_1(\xi)\, \mathbf{U}_2(x), & \xi \leqslant x \end{cases}$$

where

$$\text{(24)} \qquad \mathbf{c} = 2\, \mathbf{a}\mathbf{r}\, [\exp(x_1\, \mathbf{r}) - \exp(-x_1\, \mathbf{r})] = 2\, \mathbf{a}\mathbf{r}\, \exp(x_1\, \mathbf{r})\, [\mathbf{I} - \exp(-2\, x_1\, \mathbf{r})]$$

$$\text{(25)} \qquad \begin{aligned} \mathbf{U}_2(\xi)\, \mathbf{U}_1(x) = \exp(x_1\, \mathbf{r})\, \{&\exp[-(x_1 - \xi)\, \mathbf{r} - (x_1 - x)\, \mathbf{r}] - \exp[-(\xi - x)\, \mathbf{r}] - \\ &- \exp[-(x_1 - \xi)\, \mathbf{r} - (x_1 + x)\, \mathbf{r}] + \exp[-(\xi + x)\, \mathbf{r}]\}, \quad x \leqslant \xi; \\ \mathbf{U}_1(\xi)\, \mathbf{U}_2(x) = \exp(x_1\, \mathbf{r})\, \{&\exp[-(x_1 - \xi)\, \mathbf{r} - (x_1 - x)\, \mathbf{r}] - \exp[-(x - \xi)\, \mathbf{r}] - \\ &- \exp[-(x_1 - x)\, \mathbf{r} - (x_1 + \xi)\, \mathbf{r}] + \exp[-(x + \xi)\, \mathbf{r}]\}, \quad \xi \leqslant x. \end{aligned}$$

The unique solution of equation (2) under boundary condition (22) is:

$$\text{(26)} \qquad \mathbf{U}(x) = -\int_0^{x_1} \mathbf{G}(x, y)\, \mathbf{f}(y)\, dy, \quad x \in (0, x_1).$$

Now we can state:

T h e o r e m 3. *Let us suppose that the conditions of Theorem* 1 *and Theorem* 2 *are satisfied. Then equation (*1′*) has a unique solution given by relation (*24*) which satisfies conditions:*

$$\text{(27)} \qquad \begin{aligned} &U(x, 0) = u_0(x) \qquad &U(0, t) = 0 \\ &U_t'(x, 0) = u_1(x) \qquad &U(x_1, t) = 0. \end{aligned}$$

*This solution is always a numerical function.*

Proof. It remains to prove only the last statement.

First we analyse the expression for $\frac{1}{\mathbf{c}}$ given by relation (24). We showed that $\frac{1}{\mathbf{r}}$ is from $C$ (see relation (17)) and $e^{-x\mathbf{r}}$ is given by relation (18). Taking these facts into account, we see that the series

$$[1-\exp(-2x_1\mathbf{r})]^{-1}=\sum_{k=0}^{\infty}\exp(-2kx_1\mathbf{r})$$

is always convergent in $\mathcal{M}$ and multiplied by an element from $\mathcal{L}$ gives always an element from $\mathcal{L}$. Also it is easy to see that $\frac{1}{2\mathbf{ar}}$ is an algebraic operator of the form $\sum_{i=0}^{\infty}\delta_i l^{i/p}$.

Our solution (26) can be divided in two parts

$$\mathbf{U}(x)=\frac{1}{\mathbf{c}}\int_0^x \mathbf{U}_1(\xi)\mathbf{U}_2(x)\mathbf{f}(\xi)\,d\xi+\frac{1}{\mathbf{c}}\int_x^{x_1}\mathbf{U}_2(\xi)\mathbf{U}_1(x)\mathbf{f}(\xi)\,d\xi.$$

By Lemma 5 and its consequences it follows that $U(x)$ is an element from $\mathcal{L}(x)$.

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