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ON AFFINE STEINER TERNARY ALGEBRAS

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Abstract. In this paper, we investigate affine Steiner ternary algebras (*ASTA*'s) and give their representation by unipotent abelian groups, for which a derived loop is used. For a finite *ASTA*, the elements and quadruples of the associated Steiner quadruple system (*SAS*) are respectively the points and planes of an affine space over $GF(2)$. This implies an elementary proof of Cameron's result which states that affine geometry is characterized among *SQS*'s by a symmetric difference property.

It is well known that there is a one-to-one correspondence between the set of Steiner triple systems $S(2, 3, n)$ and the set of idempotent totally symmetric binary quasigroups (or Steiner quasigroups) of order n, i.e. quasigroups $Q(\cdot)$, $|Q|=n$, for which $x\cdot x=x$, $x\cdot y=y\cdot x$ and $x\cdot(x\cdot y)=y$ hold, for all x, y in Q ([6], [7], [4], [3]). Similarly, there is a one-to-one correspondence between the set of Steiner quadruple systems $S(3, 4, n)$ and the set of Steiner ternary algebras of order n, i.e. ternary algebras $Q(\ )$, $|Q|=n$, in which the following is valid:

$$xyz=xzy=yxz, \tag{1}$$

$$xxy=y, \tag{2}$$

$$xy(xyz)=z, \tag{3}$$

for all x, y, z in Q (see [3], [16], [14], [15]).

Let it be noted, that the spectrum of Steiner quadruple systems, and therefore of finite Steiner ternary algebras, consists of all natural numbers $n\equiv 2$ or 4 (mod 6) ([11]). Obviously, for $n=2$, there is a unique Steiner ternary algebra $Q(\ )$, $Q=\{0,1\}$, in which the ternary operation is given as follows: $000=011=101=110=0$, $001=010=100=111=1$.

In the sequel we shall write *SQS* and *STA* instead of Steiner quadruple system and Steiner ternary algebra, respectively. The variety of all *STA*'s will be denoted by $\mathcal{V}$.

It is easy to check that *STA* $Q(\ )$ is a ternary quasigroup, i.e. whenever in the equation $x_1x_2x_3=x_4$ any three of x_i, $i=1,2,3,4$ are given, then the remaining element is uniquelly determined. This quasigroup is idempotent, since $xxx=x$ is true, for all x in Q.

Let $AG(d, 2)$ be the d-dimensional affine space over $GF(2)$, $d \geqslant 2$, and recall that any three different points x, y, z uniquely determine a plane through x, y, z which is incident with exactly one more point. A point w is a fourth point of the plane through three different points x,y,z if and only if $x+y+z+w=0$ holds. Treating points and planes (quadruples of coplanar points) of $AG(d, 2)$ as points and blocks, we get a SQS $S(3,4,2^d)$, which we denote by $AG_2(d, 2)$.

Since for $AG(d, 2)$ the following proposition holds, "for any two different planes which have exactly two different points in common, the remaining four points are coplanar", it implies that an $AG_2(d,2)$ has a symmetric difference property (whenever the symmetric difference of two blocks has the same cardinality as a block, it is a block, [8]). Hence, if $\{x,y,z,v\}$ and $\{x, y,u,w\}$ are two blocks, then $\{z,v,u,w\}$ is a block too. In the associated STA, it means that $xyz=v$, $xyu=w$ implies $vwz=u$, i.e.

$$(A) \qquad (xyz)\,(xyu)\,z=u$$

holds, for all pairwise different x,y,z,u. This justifies the following definition: A Steiner ternary algebra $Q(\ )$ is called an affine Steiner ternary algebra (briefly $ASTA$) if the equality (A) is valid for any x,y,z,u in Q(compare [15]).

P r o p o s i t i o n 1. *A finite STA $Q(\ )$ is an ASTA if and only if the associated SQS has a symmetric difference property.*

P r o o f. The necessity follows immediately. On the other hand, if the associated SQS has a symmetric difference property, then (A) holds for all pairwise different x,y,z,u in Q. It is easy to verify, using axioms (1), (2), (3), that (A) is also valid if some of the elements x,y,z,u are equal.

Let it be noted that for any ternary algebra $Q(\ )$, conditions (1), (2), and (A) imply (3). Indeed, putting $u=y$ in (A), we have $(xyz)\,(xyy)\,z=y$, and therefore $(xyz)xz=y$, $xz(xzy)=y$, because of (1), (2). Hence, an equational basis for the subvariety $\mathcal{V}^a$ of the variety $\mathcal{V}$, consisting of $ASTA$'s is given by (1), (2) and (A).

In [15], a symmetric difference property is "translated" into algebraic language as the equality

$$(R) \qquad (xyz)\,(xyu)\,(zwu)=w$$

which is equivalent to our defining equality (A):

P r o p o s i t i o n 2. *A Steiner ternary algebra $Q(\ )$ is an ASTA if and only if (R) holds for all x,y,z,u,w in Q.*

P r o o f. Let $Q(\ )$ be a STA satisfying (R). Then, for $w=u$, we get $(xyz)\,(xyu)\,(zuu)=u$, which implies (A), because of (1), (2). Now if $Q(\ )$ is an $ASTA$, then (A) and $uz(zwu)=w$ hold for all x, y, z, u, w in Q, i.e. $((xyz)\,(xyu)z)\ z\,(zwu)=w$ is valid. But, by (A), (1), (3), we obtain

$$(A') \qquad (xyu)\ uz=xyz$$

for all x,y,z,u in Q, (conversely, (A'), (1), (3) imply (A)), which applied to the previous equality gives (R).

Since the equality $x+y+z+w=0$, for any four points x,y,z,w in $AG_2(d, 2)$ is equivalent to $xyz=w$ in the associated *STA*, this implies that

$$(B)\qquad (x_1 x_2 x_3)(x_4 x_5 x_6)(x_7 x_8 x_9)=(x_{p(1)} x_{p(2)} x_{p(3)})(x_{p(4)} x_{p(5)} x_{p(6)})(x_{p(7)} x_{p(8)} x_{p(9)})$$

holds for any permutation p of $\{1,2, \ldots, 9\}$ and all $x_1,x_2, \ldots x_9$ (bisymmetry law).

A Steiner ternary algebra which satisfies the bisymmetry law will be called a bisymmetric *STA* or shortly *BSTA*. As a consequence of the axiom (1) we obtain the following proposition:

P r o p o s i t i o n 3. *A Steiner ternary algebra $Q(\ )$ is a BSTA if and only if the equality*

$$(B')\qquad (x_1x_2x_3)(y_1y_2y_3)z=(y_1x_2x_3)(x_1y_2y_3)z$$

is valid for all $x_1, x_2\ x_3, y_1, y_2, y_3, z$ in Q.

C o r o l l a r y. *A Steiner ternary algebra $Q(\ )$ is a BSTA if and only if it is medial, i.e.*

$$(M)\qquad (x_1x_2x_3)(y_1y_2y_3)(z_1z_2z_3)=(x_1y_1z_1)(x_2y_2z_2)(x_3y_3z_3)$$

holds for all $x_1,x_2,x_3,y_1,y_2,y_3,z_1,z_2,z_3$ in Q.

P r o p o s i t i o n 4. *A Steiner ternary algebra $Q(\ )$ is an ASTA if and only if it is a BSTA.*

P r o o f. Let $Q(\ )$ be a *BSTA*. Then, for all x,y,z,u in Q, we get

$$(xyz)(xyu)z=(xyz)(xyu)(zzz)=(xxz)(yyz)(zuz)=zzu=u,$$

i.e. $Q(\ )$ is an *ASTA*. Conversely, let $Q(\ )$ be an *ASTA*. To show that $Q(\ )$ is bisymmetric it is enough to prove that (B') is valid for all $x_1,x_2,x_3,y_1,y_2,y_3,z$ in Q. Let $v=(x_1x_2x_3)(y_1y_2y_3)z$, which implies $x_2x_3x_1=zv(y_2y_3y_1)$. Since $Q(\ )$ is a ternary quasigroup there is an element x in Q, such that $v=(y_1x_2x_3)(xy_2y_3)z$, i.e. $x_2x_3y_1=$ $=zv(y_2y_3x)$ holds. Therefore, we obtain

$$x_1=(x_2x_3y_1)(x_2x_3x_1)y_1=(zv(y_2y_3x))(zv(y_2y_3y_1))y_1.$$

On the other hand, by Proposition 2 we get

$$x_1=(zv(y_2y_3x))(zv(y_2y_3y_1))((y_2y_3x)x_1(y_2y_3y_1)),$$

which together with the previous result implies $(y_2y_3x)x_1(y_2y_3y_1)=y_1$, i.e.

$$x_1=(y_2y_3y_1)(y_2y_3x)y_1=x.$$

Hence $(y_1x_2x_3)(x_1y_2y_3)\,z=v=(x_1x_2x_3)(y_1y_2y_3)\,z$, which was to be proved.

C o r o l l a r y. *The subvariety consisting of BSTA's coincides with the subvariety* $\mathcal{V}^a$.

Let us recall that for a Steiner binary quasigroup $Q(\cdot)$ mediality (i.e. $(x\cdot y)\cdot(z\cdot u)=(x\cdot z)\cdot(y\cdot u)$) is equivalent to $((x\cdot y)\cdot z)\cdot u=((x\cdot u)\cdot z)\cdot y$, as was shown in [10], [13]. In the ternary case, an analogoue of the mentioned identity is

$$(E)\qquad ((xy_1y_2)z_1z_2)u_1u_2=((xu_1u_2)z_1z_2)y_1y_2$$

and we shall prove its equivalence to mediality for *STA's*.

Obviously, if $Q(\ )$ is a *STA*, then (E) holds for all $x,y_1,y_2,z_1,z_2,u_1,u_2$ in Q if and only if

$$(E')\qquad (xy_1y_2)z_1z_2=(xz_1z_2)y_1y_2$$

is valid, for all x,y_1,y_2,z_1,z_2 in Q. This fact will be used in a proof of the following proposition.

P r o p o s i t i o n 5. *A Steiner ternary algebra* $Q(\ )$ *is medial if and only if it satisfies* (E) *for all* $x,y_1,y_2,z_1,z_2,u_1,u_2$ *in* Q.

P r o o f. Let $Q(\ )$ be a medial *STA*. This implies
$(xy_1y_2)z_1z_2=(xy_1y_2)\,(z_1xx)\,(z_2xx)=(xz_1z_2)\,(y_1xx)\,(y_2xx)=(xz_1z_2)y_1y_2$
and (E) follows. On the contrary, if $Q(\ )$ is a *STA* satisfying (E), and therefore (E'), we obtain

$$(xyu)uz=(uxy)\ uz=(uuz)\ xy=zxy,$$

i.e. (A') holds, implying $Q(\ )$ is an *ASTA*.

Now, we can unify our results:

T h e o r e m 1. *A Steiner ternary algebra* $Q(\ )$ *is an ASTA iff any of the identities* (A), (R), (B), (B'), (M), (E), (E') *is satisfied.*

Now, for a fixed element o in Q we define a derived binary operation $\overset{o}{+}$ by the formula

$$x\overset{o}{+}y=oxy,$$

(compare [2]), which will enable us to investigate the structure of *ASTA's*. As a consequence of the axioms (1), (2) and (3) we have the following proposition:

P r o p o s i t i o n 6. *If* $Q(\ )$ *is a STA, then* $Q(\overset{o}{+})$ *is a totally symmetric loop with an identity o.*

Hence, for a *STA* $Q(\ )$, the derived loop $Q(\overset{o}{+})$ is unipotent (any totally symmetric loop is such one), i.e. $x\overset{o}{+}x=o$ holds, for all x in Q.

Proposition 7. *If $Q(\ )$ is an ASTA, then $Q(\overset{o}{+})$ is a unipotent abelian group. Moreover, for any three elements x,y,z in Q the equality*

$$xyz = x \overset{o}{+} y \overset{o}{+} z$$

is valid.

Proof. According to the previous proposition, we have only to prove the associativity of the binary operation $\overset{o}{+}$; indeed, considering (A'), (1), (3) it follows that

$$(x \overset{o}{+} y) \overset{o}{+} z = o(x \overset{o}{+} y)z = o(oxy)z = (xyo)oz = xyz = yzx = (yzo)ox = ox(oyz) = x \overset{o}{+} (y \overset{o}{+} z),$$

for all x,y,z in Q. At the same time, we have proved the second statement of the proposition.

Corollary. *Let $Q(\ )$ be a finite ASTA, $|Q| \geqslant 2$. Then $Q(\overset{o}{+})$ is isomorphic to the elementary abelian group $Z_2 \oplus \ldots \oplus Z_2$.*

Therefore, for a finite *ASTA* the order is equal to 2^d, for some $d \in \{0,1,2,\ldots\}$. Of course, there are *STA*'s of order 2^d, for some $d > 3$, which are not *ASTA's* (one could use the examples given in [2] and [12])

Proposition 8. *If $Q(\ )$ is an ASTA, then $Q(\overset{o}{+})$ is isomorphic to $Q(\overset{o'}{+})$, for any two o, o' in Q.*

Proof. For a bijection $f: Q \to Q$, defined by the formula $f(x) = xoo'$, for x in Q, we obtain

$$f(x \overset{o}{+} y) = (xyo)oo' = xyo' = (oo'x)(oo'y)o' = f(x)f(y)o' = f(x) \overset{o'}{+} f(y),$$

by (A'), (R). This proves our statement (compare [2], Proposition 2).

Now, let us suppose $Q(+)$ is a unipotent abelian group and define a ternary operation $*$ on Q by

$$x * y * z = x + y + z, \tag{4}$$

for all x,y,z in Q. Obviously, $Q(*)$ is an *ASTA* and taking into account Proposition 7 we have a complete characterization of *ASTA's*:

Theorem 2. *If $Q(\ )$ is an ASTA, then there is a unipotent abelian group $Q(+)$, such that $Q(\ )$ is isomorphic to the ASTA $Q(*)$ defined on Q by the equality* (4).

Corollary 1. *If $Q(\ )$ is a finite ASTA, $|Q| > 2$, then in the associated SQS points and blocks are points and planes of an $AG(d, 2)$, for some $d \geqslant 2$, i.e. $Q(*)$ is isomorphic to $AG_2(d, 2)$.*

Furthermore, it means that *SQS* associated to a finite medial *STA* is in fact an $AG_2(d,2)$, similarly to the Steiner triple system associated to a finite medial Steiner quasigroup, which is an $AG_1(d,3)$, i.e. *STS* whose elements and blocks are respectively the points and triplets of colinear points of the d-dimensional affine space over $GF(3)$ (see [13]).

Corollary 2. *A Steiner quadruple system has a symmetric difference property if and only if it is isomorphic to* $AG_2(d, 2)$, *for some* $d \geqslant 2$.

The last corollary is a special case of Theorem 2 in [8]. Now, let S_2 be the previously mentioned unique two-element *STA*. It is an *ASTA* and its derived loop is abelian group Z_2. Therefore, we obtain the following two corollaries (compare with Proposition 3.3. in [15] and Corollary 4 in [2])

Corollary 3. *If* $Q(\ )$ *is a finite ASTA, then it is isomorphic to a direct product* $S_2 \otimes \ldots \otimes S_2$.

Corollary 4. *The subvariety of finite ASTA's is a minimal subvariety of* $\mathcal{V}$.

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