THE SPECTRUM OF INFINITE COMPLETE MULTIPARTITE GRAPHS

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Abstract. In this paper the spectra of infinite complete multipartite graphs are considered and a spectral characterization of such graphs is given.

1. Introduction

Throughout the paper, $G$ denotes an infinite denumerable (connected or disconnected) undirected graph, without loops or multiple edges, whose vertex set is $X = \{x_1, x_2, \ldots\}$.

According to [5], its adjacency matrix $A = (a_{ij})$ is an infinite $N \times N$ matrix, where $a_{ij} = a^{-j-i} - \frac{2}{a}$ if $x_i$ and $x_j$ are adjacent and $a_{ij} = 0$ otherwise ($a$ is a fixed constant, $0 < a < 1$).

The adjacency matrix $A$ of $G$ corresponds in a unique way, with a bounded linear selfadjoint operator $A$ in separable Hilbert space $H$, whose matrix representation in a fixed orthonormal basis $\{e_j\}^\infty_1$ is $A$.

The spectrum $\sigma(G)$ of $G$ is defined to be the spectrum $\sigma(A)$ of the operator $A$ corresponding to the adjacency matrix $A$. Since the operator $A$ is always nuclear ([3]) and consequently compact, its spectrum consists of a sequence $\lambda_1, \lambda_2, \ldots$ of non-zero eigenvalues (each appearing according to its multiplicity, which is always finite) and the zero.

We notice that the vertex set $X$ of a graph $G$ can be partitioned uniquely, into a finite or infinite number of disjoint subsets $X_1, X_2, \ldots$ so that any two vertices from the same subset are not adjacent, and any two subsets are completely adjacent or completely non-adjacent in the graph $G$.

The sets $X_1, X_2, \ldots$ are equivalence classes of an equivalence relation in $X$ defined as follows: the vertices $x$ and $y$ are equivalent iff they have the same neighbours.

The subsets $X_1, X_2, \ldots$ are called the characteristic subset of graph $G([6])$. The subgraph $g$ of $G$, obtained by choosing a fixed vertex from each of the characteristic subsets, is named the canonical image of graph $G$. 
An infinite graph $G$ is called a complete multipartite graph if any two characteristic subsets $X_i$ and $X_j$ ($i \neq j$) are completely adjacent in the graph $G$.

A complete multipartite graph $G$ having $k$ characteristic subsets $X_1, \ldots, X_k$ is denoted by $K_{X_1, \ldots, X_k}$. A complete multipartite graph $G$ which has infinitely many characteristic subsets $X_1, X_2, \ldots$ is denoted by $K_{X_1, X_2, \ldots}$.

2. On the spectrum of complete multipartite graphs

If the vertices of a complete multipartite graph $G$ are denumerated so that $X_1 = \{x_{i_1}, x_{i_2}, \ldots\}$, $X_2 = \{x_{i_3}, x_{i_4}, \ldots\}$, ..., then in the permuted basis $\{e_{i_1}, e_{i_2}, \ldots\} \cup \{e_{i_3}, e_{i_4}, \ldots\} \cup \ldots$ of the separable Hilbert space $H$, the adjacency matrix $\mathcal{A}$ of $G$ has the form:

$$\begin{bmatrix}
A_{12} & A_{13} & \cdots \\
A_{21} & 0 & A_{23} \\
A_{31} & A_{32} & 0 \\
\vdots & \vdots & \vdots 
\end{bmatrix}$$

(1)

where

$$A_{pq} = \begin{bmatrix}
a^{p+q-2} & a^{p+q-2} & \cdots \\
a^{p+q-2} & a^{p+q-2} & \cdots \\
\vdots & \vdots & \vdots 
\end{bmatrix} \quad (p = 1, 2, \ldots; q = p + 1, p + 2, \ldots)$$

and $A_{qp} = A_{pq}^T$.

Let $N_k = \{i^k_1, i^k_2, \ldots\}$ $(k = 1, 2, \ldots)$ and denote:

$$c_k = \sum_{\sum_{q}^{i^k_q}} a^{2i^k_q - 2}$$

(2)

The following theorems describe the spectra of complete multipartite graphs. At first, we consider the case when such a graph possesses infinitely many characteristic subsets.

**Theorem 1.** Let $G = K_{X_1, X_2, \ldots}$ be a complete multipartite graph with infinitely many characteristic subsets. Then its spectrum is infinite, and the next is valid:

(a) If $G \neq K_\infty$ (i.e. not each $X_i$ is a singleton), then $\lambda = 0$ is its eigenvalue; if $G = K_\infty$, then $\lambda = 0$ is not its eigenvalue.

(b) $\lambda = -c_i$ $(i = 1, 2, \ldots)$ is an eigenvalue of $G$ iff the number $c_i$ appears in the sequence $c_1, c_2, \ldots$ $p$-folds $(p > 1)$, and then its multiplicity is $p - 1$. 
(c) There is exactly one positive eigenvalue of $G$, and all other eigenvalues distinct from 0 and $-c_i$ \((i=1, 2, \ldots)\) are simple and determined by equation

\[
(3) \quad f(\lambda) = \sum_{k=1}^{\infty} \frac{c_k}{\lambda + c_k} = 1.
\]

**Proof.** Let $\lambda$ be an arbitrary eigenvalue of $G$ and $x = (x_1, x_2, \ldots)^T \neq 0$ be a corresponding eigenvector. Then from $Ax = \lambda x$ we have

\[
(4) \quad \sum_{j=1}^{\infty} a_{ij} x_j = \lambda x_i \quad (i=1, 2, \ldots).
\]

Since then the adjacency matrix of $G$ is of the form (1), relation (4) becomes

\[
(5) \quad \sum_{i_k \in N_1} a_{i_k}^{i-2} x_{i_k} + \cdots + \sum_{i_k \in N_k} a_{i_k}^{i-2} x_{i_k-1} + \sum_{i_k+1 \in N_k} a_{i_k}^{i+2} x_{i_k+1} + \cdots =
\]

\[
= \frac{\lambda}{d^k} x_p^k \quad (i_k \in N_k; \quad k=1, 2, \ldots).
\]

Consider at first the case $\lambda = 0$. Then from (5):

\[
(6) \quad \sum_{v=1}^{\infty} Y_v = Y_k \quad (k=1, 2, \ldots).
\]

where $Y_v = \sum_{i_v \in N_v} a_{i_v}^{i-2} x_{i_v}$ \((v=1, 2, \ldots)\). Since the series $\sum_{v=1}^{\infty} Y_v$ is convergent, it follows that $Y_1 = Y_2 = \cdots = 0$, i.e.

\[
\sum_{i_v \in N_v} a_{i_v}^{i-2} x_{i_v} = 0 \quad (v=1, 2, \ldots).
\]

If now $d_v$ denotes the cardinal number of $X_v(v=1, 2, \ldots)$, then $d_v=1$ implies $x_v=0$, and $d_v>1$ implies that the vector $x^v = \sum_{i_v \in N_v} x_{i_v}$ is orthogonal to the vector $a^v = \sum_{i_v \in N_v} a_{i_v}^{i-2} e_{i_v}$, where $x^v, a^v \in H_v = \mathcal{P} \{e_{i_v}^{1}, e_{i_v}^{2}, \ldots \}$. In the latter case, the vector $x^v$ forms a corresponding closed hyperplane $H^v$ of the space $H_v$. Hence, we get that, except for the case $d_v=1 \quad (v=1, 2, \ldots)$, i.e. $G=K_\infty$, $\lambda = 0$ is an eigenvalue of $G$. The corresponding proper subspace is then $H_1 \oplus H_2 \oplus \ldots$.
Now, let $\lambda \neq 0$. Then from (5) we easily find:

$$x_{ik}^k = a^{k-i} x_{i1}^k \quad (i^k \in N_k; \quad k = 1, 2, \ldots)$$

and (5) reduces to relation

$$\sum_{v=1}^{\infty} \frac{c_v}{a^{iv}} x_{iv}^v = \frac{\lambda + c_k}{a^{ik}} x_{i1}^k \quad (k = 1, 2, \ldots).$$

Since $x \neq 0$, at least one of $x_{i1}^k \quad (k = 1, 2, \ldots)$ must be non-zero. For instance $x_{i1}^1 \neq 0$. Then from (6) we find

$$\lambda c_k x_{i1}^k = a^{k-1} \lambda  + c_1 \quad x_{i1}^1 \quad (k = 2, 3, \ldots).$$

Now, since $x \neq 0$, one can conclude that $\lambda = -c_k \quad (k = 1, 2, \ldots)$ is not an eigenvalue of $G$ if $c_k$ appears in (2) exactly once.

If $c_k$ appears in (2) exactly $p$-times ($p \geq 2$), then $\lambda = -c_k$ is an eigenvalue of $G$ whose multiplicity is $p - 1$. If, for instance, $c_1 = c_2 = c_3$, then the vector $x$ (where $x_{i1}^1$ and $x_{i1}^2$ are arbitrary, $x_{i1}^1 = x_{i1}^2 = \ldots = 0$ and $x_{i1}^1$ is determined from (6)) is an eigenvector corresponding to the eigenvalue $\lambda = -c_1$.

We notice that sequence (2) cannot contain infinitely many equal elements $c_k$, because all $c_k > 0$ and

$$\sum_{k=1}^{\infty} c_k = \sum_{i=1}^{\infty} a^{2i-2} = \frac{1}{1 - a^2}.$$  

If $\lambda \neq -c_k \quad (k = 1, 2, \ldots)$, then (7) implies:

$$x_{i1}^v = \frac{a^{v-1} \lambda + c_1}{\lambda + c_v} x_{i1}^1 \quad (v = 2, 3, \ldots).$$

Now substituting $x_{i1}^v \quad (v = 2, 3, \ldots)$ into the first relation (6) we get that non-zero eigenvalues of $G$ distant from $-c_k \quad (k = 1, 2, \ldots)$ satisfy equation (3). Since the corresponding eigenvectors are uniquely determined, these eigenvalues are simple.

Besides, it is easily seen that the converse is true. Namely, if $\lambda$ is an arbitrary root of (3), then $\lambda$ is a simple eigenvalue of graph $G$.

Let all mutually distinct elements in (2) be ordered in a decreasing sequence $c_{i1}$, $c_{i2}$, $\ldots$, and let $I_v = (-c_{i_{v}}, -c_{i_{v+1}}) \quad (v = 1, 2, \ldots)$.
Then it can be shown that the functional series on the right side of (3) can be differentiated in each of the intervals \((-\infty, -c_i), I, (v=1, 2, \ldots), (0, \infty)\). We get that in all these intervals:

\[
f'(\lambda) = -\sum_{k=1}^{\infty} \frac{c_k}{(\lambda + c_k)^2} < 0.
\]

From that we conclude that the function \(f(\lambda)\) is strongly monotonically decreasing in all these intervals.

It can also be shown that if \(\psi = 1, 2, \ldots\),

\[
\lim_{\lambda \to -c_i + 0} f(\lambda) = -\infty, \quad \lim_{\lambda \to -c_i - 0} f(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda \to \pm \infty} f(\lambda) = 0.
\]

Hence, equation (3) possesses exactly one root in each of the intervals \(I, (\psi=1, 2, \ldots)\) and \((0, \infty)\).

Hence, the theorem is proved. ■

Now we consider the case when \(G\) possesses finitely many characteristic subsets.

**Theorem 2.** Let \(G = K_{x_1, \ldots, x_k}\) be a complete multipartite graph with finitely many characteristic subsets. Then

(a) \(\lambda = 0\) is an eigenvalue of \(G\);

(b) \(\lambda = -c_i\) \((i=1, \ldots, k)\) is an eigenvalue of \(G\) iff the number \(c_i\) appears in the sequence \(c_1, \ldots, c_k\) \(p\)-times \((1 < p \leq k)\), and then its multiplicity is \(p - 1\);

(c) \(G\) has exactly \(k\) non-zero eigenvalues, exactly one of which is positive. The eigenvalues distinct from \(0\) and \(-c_i\) \((i=1, \ldots, k)\) are simple and determined by the equation

\[
f(\lambda) = \sum_{i=1}^{k} \frac{c_i}{\lambda + c_i} = 1.
\]

**Proof.** Graph \(G = K_{x_1, \ldots, x_k}\) is of a finite type \(k\) and its canonical image \(g\) is a complete finite graph with \(k\) vertices. Since \(g\) has exactly \(k\) non-zero eigenvalues (taking into account their multiplicities too), graph \(G\) must have exactly \(k\) non-zero eigenvalues (see [6], Theorem 1).

Since the spectrum of a compact selfadjoint operator \(A\) is finite iff its range \(\mathcal{R}(A)\) is finite dimension, we conclude that \(\lambda = 0\) is an eigenvalue of \(G\).

The remaining part of Theorem 2 can be proved analogously to the corresponding part of Theorem 1. ■

**3. A spectral characterization of complete multipartite graphs**

We first quote two auxiliary results.

Let \(G_0\) be a (finite or infinite) induced subgraph of an infinite graph \(G\), whose vertex set is \(X_0 = \{x_{i_1}, x_{i_2}, \ldots\}\). Then its adjacency matrix \(\mathcal{A}_{G_0}\) is the
corresponding submatrix \((a_{ij})\); thus the vertices of \(G_0\) have weights \(a_{i-1}, a_{i-1}, \ldots, \) respectively.

Let \(H_0 = \overline{\text{span}} \{e_i^+, e_i^-, \ldots\}\) be the closed linear hull of elements \(e_i^+, e_i^-, \ldots\) and \(P : H \to H_0\) be the orthogonal projection of \(H\) onto the subspace \(H_0\). Then the spectrum \(\sigma(G_0)\) of \(G_0\) is defined to be the spectrum \(\sigma(A_0)\) of the operator \(A_0 = PAP|_{H_0}\). Its matrix in basis \(\{e_i^+, e_i^-, \ldots\}\) of \(H_0\) is represented by the matrix \(A_0\).

The following theorem gives a relation between the spectrum of \(G\) and any induced subgraph \(G_0\).

**Theorem 3. (Interlacing Theorem).** Let

\[
\lambda_1^+ \geq \lambda_2^+ \geq \cdots > 0; \quad \lambda_1^- \leq \lambda_2^- \leq \cdots < 0
\]

be the sequences of positive and negative eigenvalues of graph \(G\), respectively, and let

\[
\mu_1^+ \geq \mu_2^+ \geq \cdots > 0; \quad \mu_1^- \leq \mu_2^- \leq \cdots < 0
\]

be the corresponding sequences of positive and negative eigenvalues of an induced subgraph \(G_0\). Then

\[
\lambda_n^+ \geq \mu_n^+; \quad \lambda_n^- \leq \mu_n^- \quad (n = 1, 2, \ldots).
\]

**Proof.** The proposed proof is similar to that in the finite dimensional case (see [2], p. 405).

By virtue of a known theorem (see [4], p. 256) we have

\[
\lambda_n^+ = \min_{y_1, \ldots, y_{n-1} \in H} \left[ \max_{x \in H_0, ||x||=1} (Ax, x) \right], \quad \mu_n^+ = \min_{z_1, \ldots, z_{n-1} \in H_0} \left[ \max_{x \in H_0, ||x||=1} (A_0x, x) \right].
\]

Since

\[
\max_{x \in H_0, ||x||=1} (Ax, x) = \max_{x \in H_0, ||x||=1} (A_0x, x)
\]

we immediately have:

\[
\lambda_n^+ = \min_{y_1, \ldots, y_{n-1} \in H} \left[ \max_{x \in H_0, ||x||=1} (Ax, x) \right] \geq \min_{y_1, \ldots, y_{n-1} \in H} \left[ \max_{x \in H_0, ||x||=1} (A_0x, x) \right] = \mu_n^+.
\]

Further, by applying the analogous statements concerning \(\lambda_n^-\) and \(\mu_n^-\) one can show that \(\lambda_n^- \leq \mu_n^-\).

The following lemma gives an expected relation between the spectrum of (finite or infinite) graphs \(G_1, G_2, \ldots\) and their infinite union \(\bigcup_i G_i\). The uni-
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on \( \bigcup_i G_i \) of graphs \( G_1 = (X_1, U_1), G_2 = (X_2, U_2), \ldots \) \( (X_i \cap X_j = \emptyset, i \neq j) \) is the graph \( G = (X, U) \), where \( X = \bigcup_i X_i, U = \bigcup_i U_i \).

**Lemma.** The spectrum of an infinite union \( \bigcup_i G_i \) of graphs \( G_1, G_2, \ldots \) coincides with the union of their spectra \( \sigma(G_1), \sigma(G_2), \ldots \) including zero:

\[
\sigma\left( \bigcup_i G_i \right) = \left( \bigcup_i \sigma(G_i) \right) \cup \{0\}.
\]

**Proof.** If \( X_i = \{x_{i1}, x_{i2}, \ldots\} \), put \( H_i = \sum e_{ij}, e_{ji}, \ldots \). Then \( H_i \) \( (i = 1, 2, \ldots) \) are closed mutually orthogonal subspaces of \( H \) and \( H = \sum H_i \).

But each of graphs \( G_1, G_2, \ldots \) is an induced subgraph of the union. Let \( A_i \) be the adjacency matrix of \( G_i \), and \( A \) the corresponding operator on \( H \) \( (i = 1, 2, \ldots) \).

Then the subspace \( H_i \) is invariant under \( A \), i.e. \( A(H_i) \subset H_i \) and \( A|_{H_i} = A_i \) \( (i = 1, 2, \ldots) \).

Let \( \lambda \in \sigma\left( \bigcup_i G_i \right) \). Then there is an eigenvector \( x = \sum_i x_i \) \( (x_i \in H_i, i = 1, 2, \ldots) \) such that \( Ax = \lambda x \). There is some \( x_{i_0} \neq 0 \) such that \( A_{i_0} x_{i_0} = \lambda x_{i_0} \). Thus, \( \lambda \in \sigma(G_{i_0}) \), which proves \( \sigma\left( \bigcup_i G_i \right) \subset \bigcup_i \sigma(G_i) \).

The proof of the inclusion \( \bigcup \sigma(G_i) \cup \{0\} \subset \sigma\left( \bigcup_i G_i \right) \) is obvious. 

**Corollary.** The spectrum of an infinite disconnected undirected graph \( G \) coincides with the union of spectra of its connected components and zero. 

Finally, we note a theorem which gives a spectral characterization of complete multipartite graphs.

**Theorem 4.** An infinite graph \( G \) has exactly one positive eigenvalue iff its non-isolated vertices form a complete multipartite graph.

**Proof.** In virtue of the lemma, we may ignore the isolated vertices.

Primarily, by Theorems 1 and 2, we see that a complete multipartite graph possesses exactly one positive eigenvalue.

Conversely, assume that \( G \) is not a complete multipartite graph. Then there are non-adjacent vertices \( x \) and \( y \) from distinct characteristic subsets, and there is a vertex \( z \) which is, for instance, adjacent to \( y \) and non-adjacent to \( x \).

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{fig1.png}
\caption{Figure 1.}
\end{figure}
Thus, $G$ contains an induced subgraph displayed in Figure 1 (a). Since $x$ is not an isolated vertex in $G$, the graph $G$ contains at least one subgraph of the form 1 (b), 1 (c), 1 (d) as an induced subgraph. Since it can be shown that with arbitrary weights ($a^{-1}, a^{j-1}, a^{k-1}, a^{l-1}$) these graphs possess exactly two positive eigenvalues, Theorem 3 shows that graph $G$ has at least two positive eigenvalues, which is a contradiction.

This completes the proof. ■

REFERENCES


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