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THE SPECTRUM OF INFINITE COMPLETE MULTIPARTITE GRAPHS

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Abstract. In this paper the spectra of infinite complete multipartite graphs are considered and a spectral characterization of such graphs is given.

1. Introduction

Throughout the paper, G denotes an infinite denumerable (connected or disconnected) undirected graph, without loops or multiple edges, whose vertex set is $X=\{x_1, x_2, \dots\}$.

According to [5], its adjacency matrih $\mathcal{A}=(a_{ij})$ is an infinite $N\times N$ matrix, where $a_{ij}=a^{i+j-2}$ if x_i and x_j are adjacent and $a_{ij}=0$ otherwise (a is a fixed constant, $0<a<1$).

The adjacency matrix $\mathcal{A}$ of G corresponds in a unique way, with a bounded linear selfadjoint operator A in separable Hilbert space H, whose matrix representation in a fixed orthonormal basis $\{e_j\}_1^\infty$ is $\mathcal{A}$.

The spectrum $\sigma(G)$ of G is defined to be the spectrum $\sigma(A)$ of the operator A corresponding to the adjacency matrix $\mathcal{A}$. Since the operator A is always nuclear ([3]) and consequently compact, its spectrum consists of a sequence $\lambda_1, \lambda_2, \dots$ of non-zero eigenvalues (each appearing according to its multiplicity, which is always finite) and the zero.

We notice that the vertex set X of a graph G can be partitioned uniquely, into a finite or infinite number of disjoint subsets $X_1, X_2, \dots$ so that any two vertices from the same subset are not adjacent, and any two subsets are completely adjacent or completely non-adjacent in the graph G.

The sets $X_1, X_2, \dots$ are equivalence classes of an equivalence relation in X defined as follows: the vertices x and y aie equivalent iff they have the same neighbours.

The subsets $X_1, X_2, \dots$ are called the characteristic subsest of graph G([6]). The subgraph g of G, obtained by choosing a fixed vertex from each of the characteristic subsets, is named the canonical image of graph G.

An infinite graph G is called a complete multipartite graph if any two characteristic subsets X_i and X_j $(i \neq j)$ are completely adjacent in the graph G.

A complete multipartite graph G having k characteristic subsets $X_1, \ldots X_k$ is denoted by $K_{X_1, \ldots, X_k}$. A complete multipartite graph G which has infinitely many characteristic subsets $X_1, X_2, \ldots$ is denoted by $K_{X_1, X_2, \ldots}$.

2. On the spectrum of complete multipartite graphs

If the vertices of a complete multipartite graph G are denumerated so that $X_1 = \{x_{i_1^1}, x_{i_2^1}, \ldots\}$, $X_2 = \{x_{i_1^2}, x_{i_2^2}, \ldots\}, \ldots$, then in the permuted basis $\{e_{i_1^1}, e_{i_2^1}, \ldots\} \cup \{e_{i_1^2}, e_{i_2^2}, \ldots\} \cup \ldots$ of the separable Hilbert space H, the adjacency matrix $\mathcal{A}$ of G has the form:

$$\mathcal{A} = \begin{bmatrix} 0 & \mathcal{A}_{12} & \mathcal{A}_{13} & \cdots \\ \mathcal{A}_{21} & 0 & \mathcal{A}_{23} & \cdots \\ \mathcal{A}_{31} & \mathcal{A}_{32} & 0 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \tag{1}$$

where

$$\mathcal{A}_{pq} = \begin{bmatrix} a^{i_1^p + i_1^q - 2} & a^{i_1^p + i_2^q - 2} & \cdots \\ a^{i_2^p + i_1^q - 2} & a^{i_2^p + i_2^q - 2} & \cdots \\ \vdots & \vdots & \end{bmatrix} \quad \begin{matrix} (p = 1, 2, \ldots; \\ q = p+1, p+2, \ldots) \end{matrix}$$

and $\mathcal{A}_{qp} = \mathcal{A}_{pq}^T$.

Let $N_k = \{i_1^k, i_2^k, \ldots\}$ $(k = 1, 2, \ldots)$ and denote:

$$c_k = \sum_{i_q^k \in N_k} a^{2 i_q^k - 2} \tag{2}$$

The following theorems describe the spectra of complete multipartite graphs. At first, we consider the case when such a graph possesses infinitely many characteristic subsets.

Theorem 1. *Let $G = K_{X_1, X_2, \ldots}$ be a complete multipartite graph with infinitely many characteristic subsets. Then its spectrum is infinite, and the next is valid:*

(a) *If $G \neq K_\infty$ (i.e. not each X_i is a singleton), then $\lambda = 0$ is its eigenvalue; if $G = K_\infty$, then $\lambda = 0$ is not its eigenvalue.*

(b) *$\lambda = -c_i$ $(i = 1, 2, \ldots)$ is an eigenvalue of G iff the number c_i appears in the sequence $c_1, c_2, \ldots$ p-folds $(p > 1)$, and then its multiplicity is $p - 1$.*

(c) *There is exactly one positive eigenvalue of G, and all other eigenvalues distinct from* 0 *and* $-c_i$ $(i=1, 2, \ldots)$ *are simple and determined by equation*

$$f(\lambda)=\sum_{k=1}^{\infty}\frac{c_k}{\lambda+c_k}=1. \tag{3}$$

Proof. Let λ be an arbitrary eigenvalue of G and $x=(x_1, x_2, \ldots)^T\neq 0$ be a corresponding eigenvector. Then from $Ax=\lambda x$ we have

$$\sum_{j=1}^{\infty} a_{ij}x_j=\lambda x_i \qquad (i=1, 2, \ldots). \tag{4}$$

Since then the adjacency matrix of G is of the form (1), relation (4) becomes

$$\sum_{i_q^1\in N_1} a^{i_q^1-2}x_{i_q^1}+\cdots+\sum_{i_q^{k-1}\in N_{k-1}} a^{i_q^{k-1}-2}x_{i_q^{k-1}}+\sum_{i_q^{k+1}\in N_{k+1}} a^{i_q^{k+1}-2}x_{i_q^{k+1}}+\cdots= \tag{5}$$

$$=\frac{\lambda}{a^{i_p^k}}x_{i_p^k}\,(i_p^k\in N_k;\quad k=1, 2, \ldots).$$

Consider at first the case $\lambda=0$. Then from (5):

$$\sum_{\nu=1}^{\infty} Y_\nu=Y_k \qquad (k=1, 2, \ldots).$$

where $Y_\nu=\sum_{i_q^\nu\in N_\nu} a^{i_q^\nu-2}x_{i_q^\nu}$ $(\nu=1, 2, \ldots)$. Since the series $\sum_{\nu=1}^{\infty} Y_\nu$ is convergent, it follows that $Y_1=Y_2=\cdots=0$, i.e.

$$\sum_{i_q^\nu\in N_\nu} a^{i_q^\nu}x_{i_q^\nu}=0 \quad (\nu=1, 2, \ldots).$$

If now d_ν denotes the cardinal number of $X_\nu(\nu=1, 2, \ldots)$, then $d_\nu=1$ implies $x_{i_1^\nu}=0$, and $d_\nu>1$ implies that the vector $x^\nu=\sum_{i_q^\nu\in N_\nu} x_{i_q^\nu}e_{i_q^\nu}$ is orthogonal to the vector $a^\nu=\sum_{i_q^\nu\in N_\nu} a^{i_q^\nu}e_{i_q^\nu}$, where x^ν, $a^\nu\in H_\nu=\overline{\mathscr{L}\{e_{i_1^\nu}, e_{i_2^\nu}, \ldots\}}$. In the latter case, the vector x^ν forms a corresponding closed hyperplane H'_ν of the space H_ν. Hence, we get that, except for the case $d_\nu=1$ $(\nu=1, 2, \ldots)$, i.e. $G=K_\infty$, $\lambda=0$ is an eigenvalue of G. The corresponding proper subspace is then $H'_1\oplus H'_2\oplus\ldots$

Now, let $\lambda\neq 0$. Then from (5) we easily find:

$$x_{i_p^k}=a^{i_p^k-i_1^k}\,x_{i_1^k}\quad (i_p^k\in N_k;\ k=1,2,\ldots)$$

and (5) reduces to relation

$$\sum_{\nu=1}^{\infty}\frac{c_\nu}{a^{i_1^\nu}}\,x_{i_1^\nu}=\frac{\lambda+c_k}{a^{i_1^k}}\,x_{i_1^k}\quad (k=1,\ 2,\ldots). \tag{6}$$

Since $x\neq 0$, at least one of $x_{i_1^k}$ $(k=1,2,\ldots)$ must be non-zero. For instance $x_{i_1^1}\neq 0$. Then from (6) we find

$$(\lambda+c_k)x_{i_1^k}=a^{i_1^k-i_1^1}(\lambda+c_1)x_{i_1^1}\quad (k=2,3,\ldots). \tag{7}$$

Now, since $x\neq 0$, one can conclude that $\lambda=-c_k\,(k=1,2,\ldots)$ is not an eigenvalue of G if c_k appears in (2) excatly once.

If c_k appears in (2) exactly p-times $(p\geqslant 2)$, then $\lambda=-c_k$ is an eigenvalue of G whose multiplicity is $p-1$. If, for instance, $c_1=c_2=c_3$, then the vector x (where $x_{i_1^2}$ and $x_{i_1^3}$ are arbitrary, $x_{i_1^4}=x_{i_1^5}=\cdots=0$ and $x_{i_1^1}$ is determined from (6)) is an eigenvector corresponding to the eigenvalue $\lambda=-c_1$.

We notice that sequence (2) cannot contain infinitely many equal elements c_k, because all $c_k>0$ and

$$\sum_{k=1}^{\infty}c_k=\sum_{i=1}^{\infty}a^{2i-2}=\frac{1}{1-a^2}.$$

If $\lambda\neq -c_k$ $(k=1,2,\ldots)$, then (7) implies:

$$x_{i_1^\nu}=\frac{a^{i_1^\nu-i_1^1}(\lambda+c_1)}{\lambda+c_\nu}\,x_{i_1^1}\quad (\nu=2,3,\ldots).$$

Now substituting $x_{i_1^\nu}$ $(\nu=2,3,\ldots)$ into the first relation (6) we get that non-zero eigenvalues of G distanct from $-c_k$ $(k=1,2,\ldots)$ satisfy equation (3). Since the corresponding eigenvectors are uniquely determined, these eigenvalues are simple.

Besides, it is easily seen that the converse is true. Namely, if λ is an arbitrary root of (3), then λ is a simple eigenvalue of graph G.

Let all mutually distinct elements in (2) be ordered in a decreasing sequence $c_{i_1}, c_{i_2},\ldots$, and let $I_\nu=(-c_{i_\nu},\ -c_{i_{\nu+1}})$ $(\nu=1,2,\ldots)$.

Then it can be shown that the functional series on the right side of (3) can be differentiated in each of the intervals $(-\infty, -c_1)$, I_ν $(\nu=1, 2, \ldots)$, $(0, \infty)$. We get that in all these intervals:

$$f'(\lambda) = -\sum_{k=1}^{\infty} \frac{c_k}{(\lambda+c_k)^2} < 0.$$

From that we conclude that the function $f(\lambda)$ is strongly monotonically decreasing in all these intervals.

It can also be shown that if $\nu=1, 2, \ldots$,

$$\lim_{\lambda\to -c_{i_\nu}-0} f(\lambda) = -\infty, \qquad \lim_{\lambda\to -c_{i_\nu}+0} f(\lambda) = +\infty \quad \text{and} \quad \lim_{\lambda\to\pm\infty} f(\lambda) = 0.$$

Hence, equation (3) possesses exactly one root in each of the intervals I_ν $(\nu=1, 2, \ldots)$ and $(0, \infty)$.

Hence, the theorem is proved. ■

Now we consider the case when G possesses finitely many characteristic subsets.

Theorem 2. *Let $G=K_{x_1,\ldots,x_k}$ be a complete multipartite graph with finitely many characteristic subsets. Then*

(a) $\lambda=0$ *is an eigenvalue of* G;

(b) $\lambda=-c_i$ $(i=1, \ldots, k)$ *is an eigenvalue of G iff the number c_i appears in the spquence $c_1, \ldots, c_k$ p-times $(1<p\leqslant k)$, and then its multiplicity is $p-1$;*

(c) *G has exactly k non-zero eigenvalues, exactly one of which is positive. The eigenvalues distinct from* 0 *and* $-c_i$ $(i=1, \ldots, k)$ *are simple and determined by the equation*

$$f(\lambda) = \sum_{i=1}^{k} \frac{c_i}{\lambda+c_i} = 1.$$

Proof. Graph $G=K_{x_1,\ldots,x_k}$ is of a finite type k and its canonical image g is a complete finite graph with k vertices. Since g has exactly k non-zero eigenvalues (taking into account their multiplicities too), graph G must have exactly k non-zero eigenvalues (see [6], Theorem 1).

Since the spectrum of a compact selfadjoint operator A is finite iff its range $\mathcal{R}(A)$ is finite dimension, we conclude that $\lambda=0$ is an eigenvalue of G.

The remaining part of Theorem 2 can be proved analogusly to the corresponding part of Theorem 1. ■

3. A spectral characterization of complete multipartite graphs

We first quote two auxiliary results.

Let G_0 be a (finite or infinite) induced subgraph of an infinite graph G, whose vertex set is $X_0=\{x_{i_1}, x_{i_2}, \ldots\}$. Then its adjacency matrix $\mathcal{A}_0$ is the corresponding submatrix $(a_{i_p i_q})$; thus the vertices of G_0 have weights a^{i_1-1}, a^{i_2-1}, ..., respectively.

Let $H_0=\overline{\mathcal{L}\{e_{i_1}, e_{i_2}, \ldots\}}$ be the closed linear hull of elements $e_{i_1}, e_{i_2}, \ldots$ and $P: H \to H_0$ be the orthogonal projection of H onto the subspace H_0. Then the spectrum $\sigma(G_0)$ of G_0 is defined to be the spectrum $\sigma(A_0)$ of the operator $A_0=PAP|_{H_0}$. Its matrix in basis $\{e_{i_1}, e_{i_2}, \ldots\}$ of H_0 is represented by the matrix $\mathcal{A}_0$.

The following theorem gives *a* relation between the spectrum of G and any induced subgraph G_0.

Theorem 3. (*Interlacing Theorem*). *Let*

$$\lambda_1^+ \geqslant \lambda_2^+ \geqslant \cdots > 0; \quad \lambda_1^- \leqslant \lambda_2^- \leqslant \cdots < 0$$

be the sequences of positive and negative eigenvalues of graph G, respectively, and let

$$\mu_1^+ \geqslant \mu_2^+ \geqslant \cdots > 0; \quad \mu_1^- \leqslant \mu_2^- \leqslant \cdots < 0$$

be the corresponding sequences of positive and negative eigenvalues of an induced subgraph G_0. Then

$$\lambda_n^+ \geqslant \mu_n^+; \quad \lambda_n^- \leqslant \mu_n^- \qquad (n=1, 2, \ldots).$$

Proof. The proposed proof is similar to that in the finite dimensional case (see [2], p. 405).

By virtue of a known theorem (see [4], p. 256) we have

$$\lambda_n^+ = \min_{y_1, \ldots, y_{n-1} \in H} \Big[\max_{\substack{x \in H,\ \|x\|=1 \\ x \perp y_1, \ldots, y_{n-1}}} (Ax, x) \Big], \qquad \mu_n^+ = \min_{z_1, \ldots, z_{n-1} \in H_0} \Big[\max_{\substack{x \in H_0,\ \|x\|=1 \\ x \perp z_1, \ldots, z_{n-1}}} (A_0 x, x) \Big].$$

Since

$$\max_{\substack{x \in H,\ \|x\|=1 \\ x \perp y_1, \ldots, y_{n-1}}} (Ax, x) \geqslant \max_{\substack{x \in H_0,\ \|x\|=1 \\ x \perp y_1, \ldots, y_{n-1}}} (Ax, x) = \max_{\substack{x \in H_0,\ \|x\|=1 \\ x \perp Py_1, \ldots Py_{n-1}}} (A_0 x, x)$$

we immediately have:

$$\lambda_n^+ = \min_{y_1, \ldots, y_{n-1} \in H} \Big[\max_{\substack{x \in H,\ \|x\|=1 \\ x \perp y_1, \ldots, y_{n-1}}} (Ax, x) \Big] \geqslant \min_{y_1, \ldots, y_{n-1} \in H} \Big[\max_{\substack{x \in H_0,\ \|x\|=1 \\ x \perp Py_1, \ldots, Py_{n-1}}} (A_0 x, x) \Big] = \mu_n^+.$$

Further, by applying the analogous statements concerning λ_n^- and μ_n^- one can show that $\lambda_n^- \leqslant \mu_n^-$. ■

The following lemma gives an expected relation between the spectrum of (finite or infinite) graphs $G_1, G_2, \ldots$ and their infinite union $\bigcup_i G_i$. The union $\bigcup_i G_i$ of graphs $G_1=(X_1, U_1)$, $G_2=(X_2, U_2)$, ... $(X_i \cap X_j = \varnothing,\ i \neq j)$ is the graph $G=(X, U)$, where $X=\bigcup_i X_i$, $U=\bigcup_i U_i$.

Lemma. *The spectrum of an infinite union $\bigcup_i G_i$ of graphs $G_1, G_2, \ldots$ coincides with the union of their spectra $\sigma(G_1), \sigma(G_2), \ldots$ including zero:*

$$\sigma(\bigcup_i G_i)=(\bigcup_i \sigma(G_i)) \cup \{0\}.$$

Proof. If $X_i=\{x_{j_1}, x_{j_2}, \ldots\}$, put $H_i=\overline{\mathscr{L}\{e_{j_1}, e_{j_2}, \ldots\}}$. Then H_i $(i=1, 2, \ldots)$ are closed mutually orthogonal subspaces of H and $H=\sum_i \oplus H_i$.

But each of graphs $G_1, G_2, \ldots$ is an induced subgraph of the union. Let $\mathcal{A}_i$ be the adjacency matrix of G_i, and A_i the corresponding operator on H_i $(i=1, 2, \ldots)$.

Then the subspace H_i is invariant under A, i.e. $A(H_i) \subset H_i$ and $A|_{H_i}=A_i$ $(i=1, 2, \ldots)$.

Let $\lambda \in \sigma(\bigcup_i G_i)$. Then there is an eigenvector $x=\sum_i x_i$ $(x_i \in H_i;\ i=1, 2, \ldots)$ such that $Ax=\lambda x$. There is some $x_{i_0} \neq 0$ such that $A_{i_0} x_{i_0}=\lambda x_{i_0}$. Thus, $\lambda \in \sigma(G_{i_0})$, which proves $\sigma(\bigcup_i G_i) \subset \bigcup_i \sigma(G_i)$.

The proof of the inclusion $(\bigcup_i \sigma(G_i)) \cup \{0\} \subset \sigma(\bigcup_i G_i)$ is obvious. ■

Corollary. *The spectrum of an infinite disconnected undirected graph G coincides with the union of spectra of its connected components and zero.* ■

Finally, we note a theorem which gives a spectral characterization of complete multipartite graphs.

Theorem 4. *An infinite graph G has exactly one positive eigenvalue iff its non-isolated vertices form a complete multipartite graph.*

Proof. In virtue of the lemma, we may ignore the isolated vertices.

Primarily, by Theorems 1 and 2, we see that a complete multipartite graph possesses exactly one positive eigenvalue.

Conversely, assume that G is not a complete multipartite graph. Then there are non-adjacent vertices x and y from distinct characteristic subsets, and there is a vertex z which is, for instance, adjacent to y and non-adjacent to x.

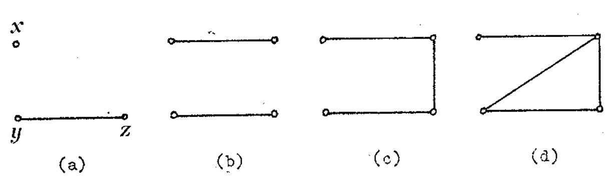

Figure 1.

Thus, G contains an induced subgraph displayed in Figure 1 (a). Since x is not an isolated vertex in G, the graph G contains at least one subgraph of the form 1 (b), 1 (c), 1 (d) as an induced subgraph. Since it can be shown that with arbitrary weights $(a^{i-1}, a^{j-1}, a^{k-1}, a^{l-1})$ these graphs possess exactly two positive eigenvalues, Theorem 3 shows that graph G has at least two positive eigenvalues, which is a contradiction.

This completes the proof. ■

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