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ABOUT AN EQUIVALENT OF THE CONTINUUM HYPOTHESIS

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The main result of this paper is the statement that every atomless ω_1-saturated Boolean algebra of power $c=2^\omega$ is isomorphic to $2^\omega/F$, where F is the Frechet filter on ω, iff $\omega_1=c$.

One part of this statement is a direct consequence of the theorem about the uniqueness of the saturated model of a complete theory. The second part is shown by constructing two ω_1-saturated Boolean algebras whose isomorphism implies CH.

As a consequence of this result we obtained the main result of [3], i.e. every Parovičenko space is homeomorphic to ω^* iff CH is true. This paper was the inspiration for our work.

The terminology used in this paper follows [1]. Basic model-theoretic notions like *complete theory*, and *k-saturated model*, are assumed to be known.

Boolean algebras are denoted by $\mathbf{A}, \mathbf{B}, \mathbf{C}\dots$, and their domains, by A, $B, C\dots$, respectively. The cardinal number of A will de denoted by $|A|$. Every Boolean algebra $\mathbf{B}$ is of the form $\mathbf{B}=(B, +, \cdot, ', \leqslant, 0, 1)$. Instead of 'Boolean algebra' we will often write BA.

Let $\mathbf{B}$ be a BA and assume $X, Y\subset B$, $a\in B$. $a<X$ stands for $\forall y\in X(a<y)$ and $a\leqslant X$ for $\forall y\in X(a\leqslant y)$. By $X<Y$ we mean $\forall x\in X\forall y\in Y(x<y)$. $X\leqslant Y$ has a similar meaning. If $x\in B$, then $a\| x$ stands for $\neg(a\leqslant x)\wedge\neg(x\leqslant a)$, and $a\| X$ for $\forall y\in X(a\| y)$. The formula $\forall x\in X\forall y\in Y(\neg(x\leqslant y))$ is denoted by $X|<Y$. The BA $\mathbf{A}$ satisfies the condition H_k iff $\mathbf{A}$ satisfies the following. Let $X, Y\subset A$, X is directed upward, Y is directed downward, $0\notin Y$, $1\notin X$, $|X|+|Y|<k$ and $X<Y$. Then there is an $a\in A$ sach that $X<a<Y$.

We will use the following statements:

Proposition 1. *An atomless BA $\mathbf{A}$ satisfies H_{ω_1} iff:*

(1) *for any $\{a_i\}_{i\in N}$, $b\in A$ such that $a_0<a_1<\cdots<a_n<\cdots<b$ there exists*

$$a\ c\in A \text{ such that } a_0<a_1<\cdots<a_n<\cdots<c<b,$$

(2) *for any increasing chain $\{a_i\}_{i\in N}$, and decreasing chain $\{b_i\}_{i\in N}$; b_i, $a_i\in A$ such that $a_0<a_1<\cdots<a_n<\cdots<\cdots<b_n<\cdots<b_1<b_0$ there exists a $c\in A$ such that*

$$a_0<a_1<\cdots<a_n<\cdots<c<\cdots<b_n<\cdots<b_1<b_0.$$

The proof of this simple statement can be found in [4] (Proposition 2.27).

P r o p o s i t i o n 2. **A** *is an atomless, k-saturated BA iff* **A** *satisfies* H_k.

This statement was proved in [4] (Theorem 2.7.).

By **2** we will denote the two-element *BA*, by 2^ω the direct product of ω copies of **2** and by $2^\omega/F$ the ultrapower modulo the Frechet filter F.

P r o p o s i t i o n 3. $2^\omega/F$ *is an* ω_1*-saturated BA, and* $|2^\omega/F|=c$.

This statement was proved by B. Jonsson, P. Olin (1968). A very elegant proof of this statemant can be found in [4] (Example 2.28.).

P r o p o s i t i o n 4. *Let* $A=\{f_{\alpha F}\,|\,\alpha<\omega_1\}$ *be a strictly decreasing chain of nonzero elements of* $2^\omega/F$. *Let* $D=\{f_F\,|\,f_{\alpha F}<f_F$ *for some* $\alpha<\omega_1\}$. *Then*:

(i) D *is a filter of* $2^\omega/F$,

(ii) $|D|=c$,

(iii) *If* $B=D\cup\{f'_F\,|\,f_F\in D\}$, *then* $\mathbf{B}=(B,\ +_B,\ \cdot_B,\ '^B,\ 0,\ 1)$ *is an* ω_1*-saturated atomless BA*,

(iv) D *is an ultrafilter of* **B**.

P r o o f. (i) Let $f_F,\ g_F\in D$ and $h\in 2^\omega$, $f_F\leqslant h_F$. Then $f_{\alpha F}<f_F$ for some $\alpha<\omega_1$, hence $f_{\alpha F}<h_F$, so that $h_F\in D$. $f_{\beta F}<g_F$ for some $\beta<\omega_1$. Let $\gamma=\max\{\alpha,\beta\}$. Then $f_{\gamma F}\leqslant f_{\alpha F}\cdot f_{\beta F}$, hence $f_{\gamma F}<f_F\cdot g_F$, i.e. $f_F\cdot g_F\in D$.

(ii) Let $f_{\alpha F}\in D$ and $f_{\alpha F}\neq 1_F$. Then $I=\{i\,|\,f(i)=0\}$ is a set of power ω. Let us order I into the sequence $I=\{i_n\,|\,n\in\omega\}$. Let F_I be the Frechet filter on I. From Proposition 3 it follows that $|2^I/F_I|=c$. Let us define for every $g\in 2^I$ an element $f_g\in 2^\omega$ such that:

$$f_g(i)=\begin{cases} f_\alpha(i), & i\notin I\\ g(i), & i\in I.\end{cases}$$

Then $f_{\alpha F}\leqslant f_{gF}$, hence $f_{gF}\in D$, and if $g_{F_I}\neq h_{F_I}$, then $f_{gF}\neq f_{hF}$. Hence $|\{f_{gF}\,|\,g\in 2^I\}|=c$, i.e. $|D|=c$.

(iii) D is closed for $+,\ \cdot$ in $2^\omega/F$, $1_F\in D$ hence **B** is a *BA*.

Let us prove that **B** is atomless. If $f_F\in D$, and $f_F\neq 0_F$ then there exists an $\alpha<\omega_1$ such that $f_{\alpha F}<f_F$ and $f_{\alpha F}\neq 0_F$. Let $g'_F\in D$. Since $2^\omega/F$ is an atomless *BA* there exists an $h_F\in D$ such that $g_F<h_F<1_F$, so that $h'_F\in B$ and $0_F<h'_F<g'_F$.

Let us prove that **B** satisfies H_{ω_1}. According to Proposition 1 it is sufficient to prove conditions (1) and (2) of Proposition 1.

We will check only condition (2). For (1) we can proceed in the same way.

Let $g_{1F}<g_{2F}<\cdots<g_{nF}<\cdots<\cdots<h_{nF}<\cdots<h_{2F}<h_{1F}$ and g_{nF}, $h_{nF}\in B$, $n\in N$.

We will consider several cases:

A) In the sequence $(g_{n_F})_{n \in N}$ there exists an $m \in N$ such that $g_{m_F} \in D$. Since $2^\omega/F$ satisfies H_{ω_1}, there exists an $f \in 2^\omega$ such that

$$g_{1F} < g_{2F} < \cdots < f_F < \cdots < h_{2F} < h_{1F}.$$

Since $g_{m_F} < f_F$ and $g_{m_F} \in D$, we have $f_F \in D$ and $f_F \in B$.

B) In the sequence $(h_{n_F})_{n \in N}$ there exists an $m \in N$ such that $h'_{m_F} \in D$. Then $h'_{2F} < h'_{2F} < \cdots < \cdots < g'_{2F} < g'_{1F}$. Since $h'_{m_F} \in D$, it follows from A that there exists an $f_F \in D$ such that $h'_{1F} < h'_{2F} < \cdots < f_F < \cdots < g'_{2F} < g'_{1F}$. Hence $g_{1F} < g_{2F} < \cdots < f'_F < \cdots < h_{2F} < h_{1F}$ and $f'_F \in B$.

C) $h_{n_F} \in D$, $g'_{n_F} \in D$, $n \in N$. Since $h_{n_F} \in D$ $n \in N$, and for every $n \in N$ there exists an $\alpha_n \in \omega_1$ such that $f_{\alpha_n F} \leqslant h_{n_F}$, and since $\{\alpha_n \mid n \in N\}$ is not cofinal with ω_1, there exists a $\beta < \omega_1$ such that $f_{\beta F} < h_{n_F}$ for every $n \in N$.

Since $2^\omega/F$ satisfies H_{ω_1} there exists $e \in 2^\omega$ such that $g_{1F} < g_{2F} < \cdots e_F < < \cdots < h_{2F} < h_{1F}$.

Let $f = e + f_\beta$. Then $g_{1F} < g_{2F} < \cdots < e_F < f_F \leqslant \cdots < h_{2F} < h_{1F}$ and $f_F \in D$.

Since the sequence $\{h_{i_F}\}_{i \in N}$ is strictly decreasing we have

$$g_{1F} < g_{2F} < \cdots < e_F < f_F < \cdots < h_{2F} < h_{1F} \text{ and } f_F \in D, \text{ hence } f_F \in B.$$

(iv) We know that D is a filter, and since for every $f_F \in B$, $f_F \in D$ or $f'_F \in D$, D is an ultrafilter.

Proposition 5. *Let* **A** *be a free BA with c generators, and E the nonprincipal ultrafilter on* ω. *Then:*

(i) $\mathbf{A}^\omega/E$ *is an* ω_1*-saturated atomles BA of power c.*

(ii) *Let p be an ultrafilter on* $\mathbf{A}^\omega/E$. *If* $q \subset p$ *and for every* $f_E \in p$ *there exists a* $g_E \in q$ *such that* $g_E \leqslant f_E$, *then* $|q| = c$.

Proof: (i) $\mathbf{A}^\omega/E$ is an ω_1-saturated BA. This statement follows from Theorem 6.1.1. of [1]. The proof that $\mathbf{A}^\omega/E$ is atomless is the same as the proof for $2^\omega/F$, or one can use Łoś's theorem. Since $|A| \leqslant |A^\omega/E| \leqslant |A^\omega|$, $|A| = c$ and $|A^\omega| = c$, we have $|A^\omega/E| = c$.

(ii) Let $G = \{a_\alpha \mid \alpha < c\}$ be set of generators for A. Let us define the set of mappings $f_\alpha \in A^\omega$, $\alpha < c$, $f_\alpha(i) = a_\alpha$, $i \in \omega$,

$$K = \{f_{\alpha E} \mid \alpha < c\} \cup \{f'_{\alpha E} \mid \alpha < c\}.$$

Let us define for every $f \in A^\omega$

$$K_f = \{f_{\alpha E} \mid \alpha < c,\ f_E \leqslant f_{\alpha E}\} \cup \{f'_{\beta E} \mid \beta < c,\ f_E \leqslant f'_{\beta E}\}.$$

Since $f(i)$ is the union of finitely many constituents of the form $a_{\alpha_1} \cdot \ldots \cdot a_{\alpha_{n(i)}} \cdot \alpha'_{\beta_1} \cdot \ldots \cdot a'_{\beta_{m(i)}}$, there are finitely manu $\alpha < \omega_1$ such that $f(i) \leqslant a_\alpha$, i.e $f(i) \leqslant f_\alpha(i)$, hence $\{f_\alpha \mid \alpha < c,\ f_E \leqslant f_{\alpha E}\}$ is a countable set. Also, for the same reason, $\{f'_\beta \mid \beta < c,\ f_E \leqslant f'_{\beta E}\}$ is a countable set.

Let p, q be as in (ii). Since for every $\alpha<c$,

$$f_\alpha\in p \text{ or } f'_\alpha\in p,\ |K\cap p|=c.$$

The conditions for p and q imply that for every $f_{\alpha E}\in p\cap K$ there exists a $g_E\in q$ such that $f_\alpha\in K_g$, and $K\cap p\subset \bigcup_{g_E\in q} K_g$.

Since $|K\cap p|=c$, we have $\left|\bigcup_{g_E\in q} K_g\right|=c$. It has already been shown that

$$|K_g|\leqslant\omega \text{ for every } g_E\in q, \text{ hence } |q|=c.$$

Theorem 6. (i) *(CH) Every atomless ω_1-saturated BA of power c is isomorphic to $2^\omega/F$.*

(ii) *If every atomless ω_1-saturated BA of power c is isomorphic to $2^\omega/F$, then $c=\omega_1$.*

Proof: (i) This statement follows directly from the completness of the theory of atomless *BA* and the uniqueness of saturated models [1].

(ii) Since **B** and $\mathbf{A}^\omega/E$ are ω_1-saturated atomless *BA* of power c, $\mathbf{B}\simeq\mathbf{A}^\omega/E$. From Proposition 4. (iv) we can see that D is an ultrafilter of B, $A\subset D$ such that $|A|=\omega_1$, and for every $f_F\in D$ there exists an $f_{\alpha F}\in A$ such that $f_{\alpha F}\leqslant f_F$. Proposition 5 (ii) and the existence of an isomorphism between **B** and $\mathbf{A}^\omega/E$ imply $|A|=c$. So $c=\omega_1$.

Corollary 7. *CH is equivalent to the statement that every Parovičenko space is isomorphic to ω^*.*

Proof: The statement directly follows from the previous theorem and the Stone Representation Theorem.

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