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MORE ON ANTI-INVERSE SEMIGROUPS

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1. In [1], [2] and [7] anti-inverse semigroups, defined as semigroups satisfying $\forall x \exists y (x = yxy,\ y = xyx)$, were studied. The following theorem was proved ([1]).

Theorem 1.1. *A semigroup S is anti-inverse iff*[1] *it satisfies*

$$\forall x \exists y (x^2 = (xy)^2 = y^2,\quad x^5 = x). \bullet$$

If $x = yxy$, $y = xyx$ call x and y mutual anti-inverses.

Let S be an anti-inverse semigroup and $x \in S$. From $x^5 = x$ we have $x = xx^3x$, $x^3xx^3 = x^7 = x^3$, $xx^3 = x^3x$, so S is completely regular, i. e. a semilattice of Rees matrix semigroups, say $S = \mathcal{S}(Y, S_\alpha)$, where Y is a semilattice, $\alpha \in Y$ and each S_α is a Rees matrix semigroup. If x and y are anti-inverses, then from $x = yxy$, $y = xyx$ it follows $x \mathcal{R} y$ and $x \mathcal{L} y$, so $x \mathcal{H} y$. Consequently, $\mathcal{H}$-classes are anti-inverse groups.

Conversely, let $S = \mathcal{S}(Y, S_\alpha)$ be a semilattice of Rees matrix semigroups over anti-inverse groups G_α, $\alpha \in Y$ and let $x \in S$. Since H_x is isomorphic to some G_α, it is anti-inverse. If y is an anti-inverse of x in H_x it is also an anti-inverse of x in S; so, S is anti-inverse.

We have just proved

Theorem 1.2. *A semigroup S is anti-inverse iff it is a semilattice of Rees matrix semigroups over anti-inverse groups.* $\bullet$

Remark. It is proved in [2] that a semigroup S is anti-inverse iff all its $\mathcal{H}$-classes are anti-inverse groups. Theorem 1.2 makes the structure of anti-inverse semigroups clearer.

Since there exists a structure theorem for completely regular semigroups (see e. g. [6]), the problem of the structure of anti-inverse semigroups is reduced to the structure of anti-inverse groups. In [2] is proved

Theorem 1.3. *A nontrivial group G is anti-inverse iff it is a union of subgroups which are isomorphic to the quaternion group or to the two-element group.* $\bullet$

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[1] „iff" stands for „if and only if".

From Theorem 1.3 we get

Corollary 1.4. *A group G is commutative and anti-inverse iff it satisfies* $x^3 = x$.

Proof. (⇒) The quaternion group is not commutative, so G, if not trivial, is covered by two-element groups. Consequently, every $x \in G$ satisfies $x^3 = x$.

(⇐) It is known that every group satisfying $x^3 = x$ is commutative. It is also anti-inverse, since x is its own anti-inverse. ●

Knowing that a semigroup is regular and commutative iff it is a semilatice of commutative groups, and using Corollary 1.4, we get

Corollary 1.5. *A semigroup S is anti-inverse and commutative iff it is a semilattice of groups satisfying* $x^3 = x$. ●

Groups satisfying $x^3 = x$ are known as Boolean groups.

2. The characterization of anti-inverse semigroups given by Theorem 1.1 is generalized in [5] and classes $\mathcal{S}_{m,n}$ of semigroups satisfying

$$(*) \qquad \forall x \exists y (x^m = (xy)^m = y^m,\ x^n = x), \qquad m, n \in N = \{1, 2, 3, \ldots\}$$

are considered. The problem investigated in [5] is which classes $\mathcal{S}_{m,n}$ are the classes of anti-inverse semigroups. We give another solution of the problem which is considerably simpler and more complete than that in [5].

By $\mathcal{A}$ ($\mathcal{G}$, $\mathcal{B}$) denote the class of anti-inverse semigroups (groups, bands).

Lemma 2.1. $\mathcal{B} \subseteq \mathcal{S}_{m,n}$, *for all m, n.*

Proof. It is easy to see that: $\mathcal{B} = \mathcal{S}_{1,2} = \mathcal{S}_{1,1}$; $\mathcal{S}_{1,1} \subseteq \mathcal{S}_{n,1}$; $\mathcal{S}_{1,2} \subseteq \mathcal{S}_{m,n}$, $(n \geq 2)$ and the assertion of the lemma follows. ●

Theorem 2.2. *If* $n \neq 1$, *then* $S \in \mathcal{S}_{m,n}$ *iff S is a semilattice of Rees matrix semigroups over groups from* $\mathcal{S}_{m,n}$.

Proof. It can be easily verified that $\mathcal{S}_{m,2} = \mathcal{B}$. But every band is a semilattice of Rees matrix semigroups over a trivial group — the only group in $\mathcal{S}_{m,2} = \mathcal{B}$.

So, let $n \geq 3$, $S \in \mathcal{S}_{m,n}$, $x \in S$. From $x^n = x$ we have: $x = xx^{n-2}x$, $x^{n-2}xx^{n-2} = x^n x^{n-3} = x^{n-2}$, $xx^{n-2} = x^{n-2}x$, and S is completely regular, i. e. a semilattice of Rees matrix semigroups. For given $x \in S$ there exists, by (*), some y such that $x^m = (xy)^m = y^m$ and therefore $x^{m+k} = y^m x^k$. For properly chosen k (say $k = n - \text{rest}\,(m, n-1)$) we will obtain $x = x^{m+k} = y^m x^k$ and similarly $y = y^{m+k} = x^m y^k$. Consequently, $x \mathcal{R} y$ and dually $x \mathcal{L} y$. Therefore, $x \mathcal{H} y$. We conclude that $\mathcal{H}$-classes are groups from $\mathcal{S}_{m,n}$.

Part of the proof in the opposite direction is similar to the same part of the proof of Theorem 1.2 and will be omitted. ●

According to the previous theorem, it is sufficient to investigate classes $\mathcal{G}_{m,n} = \mathcal{G} \cap \mathcal{S}_{m,n+1}$ of groups. Of course, we can use the language of group theory, i. e. terms as 1, x^{-5}, etc.

Theorem 2.3. $\mathcal{G}_{m,n}=\mathcal{G}_{d,n}$, *where* $d=\gcd(m,n)$ — *greatest common divisor of m and n.*

Proof. $\mathcal{G}_{d,n}\subseteq\mathcal{G}_{m,n}$, since $x^m=(xy)^d=y^d$ implies $x^m=(xy)^m=y^m$. Let us prove the opposite inclusion. If $d=\gcd(m,n)$, then there exist k, l such that $km-ln=d$ (well known fact from number theory). From $x^m=(xy)^m=y^m$ we have $x^{km}=(xy)^{km}=y^{km}$, i. e. $x^{d+ln}=(xy)^{d+ln}=y^{d+ln}$. Since $x^n=(xy)^n=y^n$, the previous equalities become $x^d=(xy)^d=y^d$. So, $\forall x\exists y(x^m=(xy)^m=y^m,\ x^n=1)$ implies $\forall x\exists y(x^d=(xy)^d=y^d,\ x^n=1)$. Consequently, we have $\mathcal{G}_{m,n}\subseteq\mathcal{G}_{d,n}$. ●

Theorem 2.3 makes it possible to make further restrictions and consider classes $\mathcal{G}_{p,pq}$ only.

Lemma 2.4. $C_r\in\mathcal{G}_{p,pq}$ *iff* r *divides* p. (C_r = *cyclic group of order* r.)

Proof. ($\Leftarrow$) Easy.

($\Rightarrow$) Let z be a generator for $C_r=\{z,z^2,\dots,z^r=1\}$. Since $C_r\in\mathcal{G}_{p,pq}$, it satisfies

$$(**)\qquad \forall x\exists y(x^p=(xy)^p=y^p)$$

Every $x\in C_r$ is of the form z^a, $1\leqslant a\leqslant r$, and condition (**), applied to C_r, gives $(\forall a\leqslant r)(\exists b\leqslant r)(z^{ap}=z^{(a+b)p}=z^{bp})$, wherefrom $(\forall a\leqslant r)(\exists b\leqslant r)(z^{ap+bp}=z^{bp})$, i. e. $(\forall a\leqslant r)(z^{ap}=1)$. This implies $(\forall a\leqslant r)$ (r divides ap) and we conclude that r divides p. ●

Corollary 2.5. $\mathcal{G}_{p,pq}\subseteq\mathcal{A}$ *iff* $p\leqslant 2$. *Moreover*

(i) $\mathcal{G}_{1,q}$ *contains the trivial group only.*

(ii) $\mathcal{G}_{2,2q}$, $q\in 2N-1$, *is the class of Boolean groups.*

(iii) $\mathcal{G}_{2,4q}=\mathcal{A}\cap\mathcal{G}$, $q\in N$.

Proof. ($\Rightarrow$) $\mathcal{G}_{p,pq}$ contains C_p by Lemma 2.4. C_p is not anti-inverse for $p>2$, so $p\leqslant 2$.

($\Leftarrow$) For $p=2$, $\mathcal{G}_{2,2q}\subseteq\mathcal{A}$, since $x^2=(xy)^2=y^2$ implies $x=yxy$, $y=xyx$; for $p=1$ is $\mathcal{G}_{1,q}\subseteq\mathcal{G}_{2,2q}\subseteq\mathcal{A}$.

(i) Groups from $\mathcal{G}_{1,q}$ satisfy $\forall x\exists y(x=xy=y)$. That implies $\forall x(x^2=x)$, i. e. $\forall x(x=1)$.

(ii) Obviously, every group satisfying $x^2=1$ is in $\mathcal{G}_{2,2q}$. Conversely, let $G\in\mathcal{G}_{2,2q}$. We have seen that G is anti-inverse, so it satisfies $x^4=1$, by Theorem 1.1 applied to groups. From $x^4=1$ and $x^{2q}=1$ we can get $x^{\gcd(4,2q)}=1$; but $\gcd(4,2q)=2$ since q is odd. Consequently, every $G\in\mathcal{G}_{2,2q}$ satisfies $(\forall x)(x^2=1)$.

(iii) The proof goes along the same lines as that in (ii). ●

Let us return to classes $\mathcal{S}_{m,n}$. What can we say about classes $\mathcal{S}_{m,1}$? We have already seen that $\mathcal{S}_{1,1}=\mathcal{B}$. $\mathcal{S}_{m,1}$ ($m\geqslant 2$) contains cyclic group C_m and also all constant semigroups (semigroups satisfying $xy=uv$), so it is not a class of anti-inverse semigroups.

Theorem 2.6. (1) $\mathcal{B}\subseteq\mathcal{S}_{m,n}$ *for all* $m, n\geqslant 1$.

(2) *Let* $n>1$. *Then* $\mathcal{S}_{m,n}\subseteq\mathcal{A}$ *iff* $\gcd(m, n-1)\leqslant 2$. *Moreover*

(2i) $\mathcal{S}_{m,n}=\mathcal{B}$ *iff* $\gcd(m, n-1)=1$.

(2ii) $\mathcal{S}_{m,n}$ *is the class of anti-inverse semigroups satisfying* $x^3=x$ *iff* $\gcd(m, n-1)=2$ *and* $n\in 4N-1$.

(2iii) $\mathcal{S}_{m,n}=\mathcal{A}$ *iff* $\gcd(m, n-1)=2$ *and* $n\in 4N+1$.

(3) $\mathcal{S}_{1,1}=\mathcal{B}$; *if* $m>1$ *then* $\mathcal{S}_{m,1}$ *is not a class of anti-inverse semigroups.*

Proof (1) Lemma 2.1.

(2) Every $S\in\mathcal{S}_{m,n}$ is a semilattice of Rees matrix semigroups over groups from $\mathcal{S}_{m,n}$ by Theorem 2.2. According to Theorem 2.3, all these groups are in $\mathcal{G}_{\gcd(m,n),n-1}$, so Corollary 2.5 can be applied, and the assertion follows.

(3) See brief considerations after Corollary 2.5. ●

3. In [1] anti-inverse semigroups are also characterized as semigroups satisfying $\forall x\exists y(x^2=y^2,\ yx=x^3y,\ x^5=x)$. This is generalized in [3], [4] and some properties of classes $\mathcal{S}^*_{m,n}$ of semigroups satisfying $\forall x\exists y(x^m=y^m,\ yx=x^{m+1}y,\ x^n=x)$ are studied.

Similar considerations, as for $\mathcal{S}_{m,n}$ above, can be applied to $\mathcal{S}^*_{m,n}$, obtaining similar results that can be proved in a similar way. This is the reason why we will omit the proofs and just state corresponding results.

At first, Lemma 2.1 and Theorem 2.2 remain true if $\mathcal{S}_{m,n}$ is replaced by $\mathcal{S}^*_{m,n}$. So, further investigations can be restricted to classes $\mathcal{G}^*_{m,n}=\mathcal{G}\cap\mathcal{S}^*_{m,n+1}$ of groups.

Lemma 3.4. $C_r\in\mathcal{G}^*_{m,n}$ *iff* r *divides* $\gcd(m, n)$. ●

Corollary 3.5. *Let* $n\neq 1$. *Then* $\mathcal{G}^*_{m,n}\subseteq\mathcal{A}$ *iff* $\gcd(m, n)\leqslant 2$.

Moreover: (i) *If* $\gcd(m, n)=1$ *then* $\mathcal{G}^*_{m,n}$ *contains the trivial group only.*

(ii) *If* $\gcd(m, n)=2$, $n\in 4N-2$ *then* $\mathcal{G}^*_{m,n}$ *is the class of Boolean groups.*

(iii) *If* $\gcd(m, n)=2$, $n\in 4N$ *then* $\mathcal{G}^*_{m,n}=\mathcal{G}\cap\mathcal{A}$. ●

Theorem 2.6 also remains true if $\mathcal{S}_{m,n}$ is replaced by $\mathcal{S}^*_{m,n}$ except in the case of $m=n=1$. In this case $\mathcal{S}_{1,1}=\mathcal{B}$, while $\mathcal{S}^*_{1,1}$ is the class of semigroups satisfying $x^3=x^2$ and properly includes $\mathcal{B}$.

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