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ON SOME INEQUALITIES OF D. C. BARNES

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1. Introduction. In this paper we shall give some generalisations of Barnes's inequalities from [1] and [2].

Let

$$H_{q_1,\ldots,q_m}(a_1,\ldots,a_m;p)=\frac{\sum_{i=1}^{n} p_i a_{1i}\ldots a_{mi}}{\prod_{j=1}^{m}\left(\sum_{i=1}^{n} p_i a_{ji}{}^{q_j}\right)^{1/q_j}},$$

and

$$H_{q_1,\ldots,q_m}(f_1,\ldots,f_m;p)=\frac{\int_a^b p(x)f_1(x)\ldots f_m(x)\,dx}{\prod_{j=1}^{m}\left(\int_a^b p(x)f_j(x)^{q_j}\,dx\right)^{1/q_j}}.$$

For an arbitrary sequence p, we shall say $a \underset{p}{\prec} b$, or equivalently $b \underset{p}{\succ} a$, if we have

$$\sum_{i=1}^{k} p_i a_i \leqslant \sum_{i=1}^{k} p_i b_i \quad (k=1,\ldots,n-1) \text{ and } \sum_{i=1}^{n} p_i a_i = \sum_{i=1}^{n} p_i b_i.$$

By analogy $f \underset{p}{\prec} g$ $(g \underset{p}{\succ} f)$ if we have, for an arbitrary function p

$$\int_a^x p(t)f(t)\,dt \leqslant \int_a^x p(t)g(t)\,dt, \ (\forall\, x\in(a,b)), \ \int_a^b p(t)f(t)\,dt=\int_a^b p(t)g(t)\,dt.$$

Let $p_i=1$ $(i=1,\ldots,n)$ and $p(x)=1$, $(\forall\, x\in[a,b])$. In this case D. C. Barnes proved the following two theorems:

Theorem A. *Let a, b be sequences having nonnegative components with $\left(\sum_{i=1}^{n} a_i{}^p\right)^{1/q}$, $\left(\sum_{i=1}^{n} a_i{}^q\right)^{1/q}>0$ where p and q are real numbers. Then*

(i) *If* $a_0=(a_{01},\ldots,a_{0n})$, $b_0=(b_{01},\ldots,b_{0n})$ *with* a_0 *increasing and* b_0 *decreassing and* $a_0 \underset{1}{\prec} a^+$, $b_0 \underset{1}{\succ} b^-$, *where* b^- *denotes the rearrangement of* b *into decreasing order, and* a^+ *denotes the rearrangement of* a *into increasing order, then*

$$H_{p,q}(a, b;\, 1)\geqslant H_{p,q}(a_0, b_0;\, 1) \quad for \quad p, q\geqslant 1.$$

(ii) *However if* a_0 *and* b_0 *are increasing with* $a^+ \underset{1}{\succ} a_0$ *and* $b^+ \underset{1}{\succ} b_0$, *then*

$$H_{p,q}(a, b;\, 1)\leqslant H_{p,q}(a_0, b_0;\, 1) \quad for \quad p, q\leqslant 1.$$

Theorem B. *Let* f, g *be nonegative functions on* $[0, a]$ *with*

$$0<\Big(\int_0^a f(x)^p\,dx\Big)^{1/p},\ \Big(\int_0^a g(x)^q\,dx\Big)^{1/q}<+\infty.\ Then$$

(i) *If* f_0 *is increasing with* $f_0 \underset{1}{\prec} f^+$ *and* g_0 *is decreasing with* $g_0 \underset{1}{\succ} g^-$, *where* g^- *denotes the rearrangement of* g *into decreasing order ang* f^+ *denotes the rearrangement of* f *into increasing order, then*

$$H_{p,q}(f, g;\, 1)\geqslant H_{p,q}(f_0, g_0;\, 1) \quad for \quad p, q\geqslant 1.$$

(ii) *If* f_0 *and* g_0 *are both increasing functions with* $f_0 \underset{1}{\prec} f^+$ *and* $g_0 \underset{1}{\prec} g^+$ *it follows then that*

$$H_{p,q}(f, g;\, 1)\leqslant H_{p,q}(f_0, g_0;\, 1) \quad for \quad p, q\leqslant 1.$$

2. Generalizations. L. Fuchs [3] (see also [4]) proved the following theorem:

Theorem C. *Let* a, b *and* p *are real sequences such that* $a_i, b_i\in I$ $(i=1,\ldots,n)$ *and* $a \underset{p}{\prec} b$. *If* a *and* b *are nonincreasing then for every convex function* $\Phi: I\to R$ *it holds that*

$$\sum_{i=1}^{n} p_i\,\Phi(a_i)\leqslant \sum_{i=1}^{n} p_i\,\Phi(b_i). \tag{1}$$

If a *and* b *are nondecreasing,* (1) *is reversed.*

By analogy we can get:

Theorem D. *Let* f, g *and* p *be real functions such that* $f(x), g(x)\in I$ $(\forall x\in[a, b])$ *and* $f \underset{p}{\prec} g$. *If* f *and* g *are nonincreasing then for every convex function* $\Phi: I\to R$

$$\int_a^b p(t)\,\Phi(f(t))\,dt\leqslant \int_a^b p(t)\,\Phi(g(t))\,dt. \tag{2}$$

If f *and* g *are nondecreasing,* (2) *is reversed.*

Suppose that all sums and integrals are positive and finite. Let

$$\|a\|_q=\left(\sum_{i=1}^{n} p_i a_i^q\right)^{1/q} \quad \text{and} \quad (a, b)=\sum_{i=1}^{n} p_i a_i b_i.$$

Then, from Theorem C we obtain

$$\|a\|_q \leqslant \|\alpha\|_q, \tag{3}$$

if sequences a and α are nondecreasing with $a \underset{p}{\succ} \alpha$, and $q \leqslant 1$ or a and α are nonincreasing, with $a \underset{p}{\prec} \alpha$ and $q \geqslant 1$.

By using Abel's identity:

$$\sum_{i=1}^{n} p_i a_i = a_n \sum_{i=1}^{n} p_i + \sum_{k=1}^{n-1}\left(\sum_{i=1}^{k} p_i\right)(a_k - a_{k+1}), \tag{4}$$

we easily get $(a, b) \geqslant (a, \beta)$, either $b \underset{p}{\prec} \beta$ and a is nondecreasing or $b \underset{p}{\succ} \beta$ and a is nonicreasing; and $(a, \beta) \geqslant (\alpha, \beta)$ either $a \underset{p}{\prec} \alpha$ and β is nondecreasing or $a \underset{p}{\succ} \alpha$ and β is nonincreasing.

Using the above results we can formulate the following theorem:

T h e o r e m 1. *Let p be a real sequence and let a, b, α, β, be nonnegative monotonous sequences. The inequality $H_{q,q'}(a, b; p) \geqslant H_{q,q'}(\alpha, \beta; p)$ holds in the following cases:*

(i) *a, b, α, β are nondecreasing with $a \underset{p}{\prec} \alpha$, $b \underset{p}{\prec} \beta$ and $q, q' \leqslant 1$;*

(ii) *a, α are nondecreasing with $a \underset{p}{\succ} \alpha$, and b, β are nonincreasing with $b \underset{p}{\prec} \beta$, and $q, q' \geqslant 1$;*

(iii) *a, b, α, β are nonincreasing with $b \underset{p}{\succ} \beta$, $a \underset{p}{\succ} \alpha$ and $q, q' \leqslant 1$.*

By analogy we can prove:

T h e o r e m 2. *Let p be a real function and let f, g, φ, γ be monotonous nonnegative functions. The inequality $H_{q,q'}(f, g; p) \geqslant H_{q,q'}(\varphi, \gamma; p)$ holds in the following cases:*

(i) *f, g, φ, γ are nondecreasing with $f \underset{p}{\prec} \varphi$, $g \underset{p}{\prec} \gamma$ and $q, q' \leqslant 1$;*

(ii) *f, φ are nondecreasing and g, γ are nonincreasing with $f \underset{p}{\succ} \varphi$, $g \underset{p}{\prec} \gamma$ and $q, q' \geqslant 1$;*

(iii) *f, g, φ, γ are nonincreasing with $f \underset{p}{\succ} \varphi$, $g \underset{p}{\succ} \gamma$ and $q, q' \leqslant 1$.*

Now, we shall prove:

T h e o r e m 3. *Let p be a real sequence and let $a_1, \ldots, a_m$, $b_1, \ldots, b_m$ be nonnegative monotonous sequences and let $q_i \leqslant 1$ $(i=1, \ldots, m)$.*

The inequality

$$H_{q_1,\dots,q_m}(a_1,\dots,a_m;p)\geqslant H_{q_1,\dots,q_m}(b_1,\dots,b_m;p) \tag{5}$$

holds either for $a_1,\dots,a_m,b_1,\dots,b_m$ *nondecreasing with* $a_i \underset{p}{\prec} b_i$ $(i=1,\dots,m)$, *or nonincreasing with* $a_i \underset{p}{\succ} b_i$ $(i=1,\dots,m)$.

Proof. Let $(a_1,\dots,a_m)=\sum_{i=1}^{n} p_i a_{1i}\dots a_{mi}$. Using (4) we have

$$(a_1,\dots,a_m)\geqslant(a_1,\dots,a_{m-1},b_m)\geqslant\cdots\geqslant(b_1,\dots,b_m),$$

either $a_1,\dots,a_m,b_1,\dots,b_m$ are nonnegative nondecreasing with $a_i \underset{p}{\prec} b_i$ $(i=1,\dots,m)$, or nonnegative nonincreasing with $a_i \underset{p}{\succ} b_i$ $(i=1,\dots,m)$. Then, according to (3), we get (5).

Theorem 4. *Let p be a real function and let $f_1,\dots,f_m,g_1,\dots,g_m$ be nonnegative monotonous functions and let $q_i\leqslant 1$ $(i=1,\dots,m)$. The inequality $H_{q_1,\dots,q_m}(f_1,\dots,f_m,p)\geqslant H_{q_1,\dots,q_m}(g_1,\dots,g_m;p)$ holds either if $f_1,\dots,f_m$, $g_1,\dots,g_m$ are nondecreasing with $f_i \underset{p}{\prec} g_i$ $(i=1,\dots,m)$ or nonincreasing with $f_i \underset{p}{\succ} g_i$ $(i=1,\dots,m)$.*

3. Applications. In this paper we shall give some applications for sums. Analogus results for integrals hold too.

First we shall prove the following lemma:

Lemma: *Let x and p be real sequences with $\sum_{i=1}^{n} p_i\geqslant 0$. The inequalities*

$$\sum_{i=1}^{k} p_i \sum_{i=1}^{n} p_i x_i \geqslant \sum_{i=1}^{n} p_i \sum_{i=1}^{k} p_i x_i \quad (k=1,\dots,n-1) \tag{6}$$

hold for every nondecreasing sequence x if and only if

$$0\leqslant P_k\leqslant P_n \text{ for } k=1,\dots,n-1 \quad \left(P_k=\sum_{i=1}^{k} p_i,\ k=1,\dots,n\right), \tag{7}$$

In the same way the reverse inequalities are satisfied in (6) for every nonincreasing sequence x.

Proof. Let $x_i=0$ $(i=1,\dots,k)$, $x_i=1$ $(i=k+1,\dots,n)$ for $k=1,\dots,n-1$. Then (6) becomes: $P_k\bar{P}_{k+1}\geqslant 0$ $(k=1,\dots,n-1)$. where $\bar{P}_{k+1}=P_n-P_k$. Since $P_n=P_k+\bar{P}_{k+1}\geqslant 0$, we have $P_k\geqslant 0$ and $\bar{P}_{k+1}\geqslant 0$ $(k=1,\dots,n-1)$, which is equivalent to (7). So, the condition (7) is necessary. Let us now prove that it is also sufficient.

By using indentity (see [5]):

$$\sum_{i=1}^{k} p_i \sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i \sum_{i=1}^{k} p_i x_i = \bar{P}_{k+1} \sum_{r=1}^{k-1} P_r (x_{r+1} - x_r) + P_k \sum_{r=k+1}^{n} \bar{P}_r (x_r - x_{r-1})$$

according to (7), we get (6).

Corollary 1. *Let* $a_1, \dots, a_m$ *be nonnegative sequences monotonous in the same sense and let* p *be a real sequence such that* (7) *and* $P_n > 0$ *hold. If* $q_1, \dots, q_m \leqslant 1$, *then*

$$\left(\sum_{i=1}^{n} p_i\right)^{\sum_{i=1}^{m} 1/q_i - 1} \sum_{i=1}^{n} p_i a_{1i} \dots a_{mi} \geqslant \prod_{j=1}^{m} \left(\sum_{i=1}^{n} p_i a_{ji}^{q_j}\right)^{1/q_j}. \tag{8}$$

Proof. Suppose that a_j $(j=1, \dots, m)$ are nondecreasing sequences. Let the sequences $b_j = (b_{j1}, \dots, b_{jn})$ $(j=1, \dots, m)$ be given by

$$b_{ji} = \frac{1}{P_n} \sum_{i=1}^{n} p_i a_{ji} \quad (i=1, \dots, n;\ j=1, \dots, m).$$

Then, using the Lemma, we have $a_j \underset{p}{\prec} b_j$ $(j=1, \dots, m)$ and from (5) we obtain (8). Analogously, we can prove (8) if a_j $(j=1, \dots, m)$ are nonincreasing sequences.

If sequence a satisfies the condition

$$a_{i-1} - 2a_i + a_{i+1} \leqslant 0 \quad (i=2, \dots, n-1), \tag{9}$$

then a is concave. If the reverse inequality in (9) holds, then a is convex. For a concave sequence a, we have that sequences (see [6] or [7])

$$A = (a_k - a_1)/(k-1)_{k=2, \dots, n} \quad \text{and} \quad B = (a_n - a_k)/(n-k)_{k=1, \dots, n-1}$$

are nonincreasing. If $a_i \geqslant 0$ $(i=1, \dots, n)$ then a sequence $A' = a_k/(n-k)_{k=2, \dots, n}$ is nonincreasing and $B' = a_k/(n-k)_{k=1, \dots, n-1}$ is nondecreasing. For convex sequence a, sequences A and B are nondecreasing. If $a_1 = 0$, sequence A' is nondecreasing and if $b_n = 0$, sequence B' is nonincreasing.

Corollary 2. *Let a, b and p be nonnegative sequences, and* $q, q' \geqslant 1$.

(i) *If a is nondecreasing concave, b is nonincreasing concave then*

$$H_{q,q'}(a, b; p) \geqslant H_{q,q'}(I_n, J_n; p) \tag{10}$$

where $I_n = (0, 1, \dots, n-1)$ *and* $J_n = (n-1, \dots, 1, 0)$.

(ii) *If a is nondecreasing convex with* $a_1 = 0$, *b is nonincreasing convex with* $b_n = 0$, *then the reverse inequality holds.*

Proof. (i) Let a and b be nondecreasing concave and nonincreasing concave respectively, and let

$$\alpha_i = \frac{\sum_{j=1}^{n} p_j a_j}{\sum_{j=1}^{n} p_j (j-1)} (i-1), \quad \beta_i = \frac{\sum_{j=1}^{n} p_j b_j}{\sum_{j=1}^{n} p_j (n-j)} (n-i) \quad (i=1, \dots, n).$$

Using the substitutions $p_i \to p_i(i-1)$, $x_i \to a_i/(i-1)$ $(i=2, \ldots, n)$, $p_1 \to p_1 \varepsilon$, $x_1 \to a_1/\varepsilon$, where ε is a positive number such that $\varepsilon \leqslant a_1/a_2$. Then, from (6) in the case when $\varepsilon \to 0$, we have

$$\sum_{i=1}^{k} p_i(i-1) \sum_{i=1}^{n} p_i a_i \leqslant \sum_{i=1}^{n} p_i(i-1) \sum_{i=1}^{k} p_i a_i$$

i.e. $\alpha \underset{p}{\prec} a$. Analogously we can prove that $\beta \underset{p}{\succ} b$. So, from Theorem 1 (ii) we obtain (10).

The proof of (ii) is similar and will not be given.

Corollary 3. *Let a_j $(j=1, \ldots, m)$ be nonnegative nondecreasing sequences, p is a nonnegative sequence, and let $0 < q_j \leqslant 1$ $(j=1, \ldots, m)$.*

(i) *If a_j $(j=1, \ldots, m)$ are concave, then*

$$H_{q_1, \ldots, q_m}(a_1, \ldots, a_m;\, p) \leqslant H_{q, \ldots, q_m}(I_n, \ldots, I_n;\, p).$$

(ii) *If a_j $(a=1, \ldots, m)$ are convex with $a_{j1}=0$ $(j=1, \ldots, m)$ then the reverse inequality holds.*

Remark. Carollaries 2 and 3 are generalizations of results from [1], [2], [6], [7] and [8].

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