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INTEGRABILITY CONDITIONS OF DERIVATIONAL FORMULAS OF A SUBSPACE OF A GENERALIZED RIEMANNIAN SPACE*)

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Summary. In the works [3] and [4] we introduced 20 Ricci-type identities of a subspace of a non-symmetric affine connexion space. In the work [5] we obtained 4 kinds of derivational formulas of a subspace of a generalized Riemannian space. Using the results of the above mentioned works, especially those identities in which curvature tensors appear (6 cases), we obtain integrability conditions of derivational formulas. From here we obtain generalizations of the Gauss and Codazzi equations.

0. Introduction

The present work is a continuation of the work [5], where we introduced derivational formulas of a subspace V_M of a generalized Riemannian space V_N. In this work the notation is in accordance with [5]. The basic tensor in the space is $a_{\alpha\beta}$ (non-symmetric) and in the subspace g_{ij} (also non-symmetric), and the subspace is given by the equations

$$y^\alpha = y^\alpha (x^1, \ldots, x^M). \tag{1}$$

Let us note that the Greek indices refer to the space and take the values $1, \ldots, N$ (except indices in brackets, which take the values from $M+1$ to N) and Latin indices refer to the subspace and take the values from 1 to M.

If we use, according to $(9a-d)$ in [5], 4 kinds of covariant derivation, we obtain 4 kinds of derivational formulas (see (16) and (37′) in [5]):

$$t^\alpha_{\underset{\theta}{i|}m} = \underset{\theta}{\Phi}{}^p_{im}\, t^\alpha_p + \sum_{\rho=M+1}^{N} \underset{\theta}{\Omega}{}_{(\rho)im} N^\alpha_{(\rho)}, \tag{2a}$$

$$N^\alpha_{\underset{\theta}{(\sigma)|}m} = -e_{(\sigma)}\, g^{ps}_{\underset{\theta}{\Omega}}{}_{(\sigma)sm}\, t^\alpha_p + \sum_{\rho=M+1}^{N} \underset{\theta}{\Psi}{}_{(\rho\sigma);m} N^\alpha_{(\rho)}, \quad \underset{\theta}{\Psi}{}_{(\sigma\sigma)m} = 0, \tag{2b}$$

$(\theta = 1, \ldots, 4)$

where $e_{(\sigma)} = \pm 1$ and $t^\alpha_i = y^\alpha_{,i} = \dfrac{\partial y^\alpha}{\partial x^i}$.

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From (28a, b) in [5] we conclude that

$$N^{\alpha}_{(\sigma)\underset{1}{|}m}=N^{\alpha}_{(\sigma)\underset{3}{|}m},\quad N^{\alpha}_{(\sigma)\underset{2}{|}m}=N^{\alpha}_{(\sigma)\underset{4}{|}m}, \tag{3a,b}$$

and

$$\underset{1}{\Omega}_{(\sigma)ij}=\underset{3}{\Omega}_{(\sigma)ij},\quad \underset{2}{\Omega}_{(\sigma)ij}=\underset{4}{\Omega}_{(\sigma)ij}, \tag{3'a, b}$$

$$\underset{1}{\Psi}_{(\rho\sigma)m}=\underset{3}{\Psi}_{(\rho\sigma)m},\quad \underset{2}{\Psi}_{(\rho\sigma)m}=\underset{4}{\Psi}_{(\rho\sigma)m}. \tag{3''a, b}$$

1. Integrability conditions of the first derivational formula

1.0. Using the Ricci-type identities (7), (11), (56) from [3] and (12), (13), (46) from [4], for the tensor t_i^{α} we obtain

$$t^{\alpha}_{i\underset{1}{|}mn}-t^{\alpha}_{i\underset{1}{|}nm}=R^{\alpha}_{\pi\mu\nu}\,t^{\mu}_m\,t^{\nu}_n\,t^{\pi}_i-\underset{1}{R}{}^{p}_{imn}\,t^{\alpha}_p-2\,\Gamma^{s}_{\underset{\vee}{mn}}\,t^{\alpha}_{i\underset{1}{|}s}, \tag{4a}$$

$$t^{\alpha}_{i\underset{2}{|}mn}-t^{\alpha}_{i\underset{2}{|}nm}=\underset{2}{R}{}^{\alpha}_{\pi\mu\nu}\,t^{\mu}_m\,t^{\nu}_n\,t^{\pi}_i-\underset{2}{R}{}^{p}_{imn}\,t^{\alpha}_p+2\,\Gamma^{s}_{\underset{\vee}{mn}}\,t^{\alpha}_{i\underset{2}{|}s}, \tag{4b}$$

$$t^{\alpha}_{i\underset{1}{|}m\underset{2}{|}n}-t^{\alpha}_{i\underset{2}{|}n\underset{1}{|}m}=\underset{3}{R}{}^{\alpha}_{\pi mn}\,t^{\pi}_i-\underset{3}{R}{}^{p}_{imn}\,t^{\alpha}_p, \tag{4c}$$

$$t^{\alpha}_{i\underset{3}{|}mn}-t^{\alpha}_{i\underset{3}{|}nm}=\underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\,t^{\mu}_m\,t^{\nu}_n\,t^{\pi}_i-\underset{2}{R}{}^{p}_{imn}\,t^{\alpha}_p+2\,\Gamma^{s}_{\underset{\vee}{mn}}\,t^{\alpha}_{i\underset{3}{|}s}, \tag{4d}$$

$$t^{\alpha}_{i\underset{2}{|}mn}-t^{\alpha}_{i\underset{4}{|}nm}=\underset{2}{R}{}^{\alpha}_{\pi\mu\nu}\,t^{\mu}_m\,t^{\nu}_n\,t^{\pi}_i-\underset{1}{R}{}^{p}_{imn}\,t^{\alpha}_p-2\,\Gamma^{s}_{\underset{\vee}{mn}}\,t^{\alpha}_{i\underset{4}{|}s}, \tag{4e}$$

$$t^{\alpha}_{i\underset{3}{|}m\underset{4}{|}n}-t^{\alpha}_{i\underset{4}{|}n\underset{3}{|}m}=\underset{4}{R}{}^{\alpha}_{\pi mn}\,t^{\pi}_i+\underset{3}{R}{}^{p}_{inm}\,t^{\alpha}_p, \tag{4f}$$

where

$$\underset{1}{R}{}^{i}_{jmn}=\Gamma^{i}_{jm,n}-\Gamma^{i}_{jn,m}+\Gamma^{p}_{jm}\,\Gamma^{i}_{pn}-\Gamma^{p}_{jn}\,\Gamma^{i}_{pm}, \tag{5a}$$

$$\underset{2}{R}{}^{i}_{jmn}=\Gamma^{i}_{mj,n}-\Gamma^{i}_{nj,m}+\Gamma^{p}_{mj}\,\Gamma^{i}_{np}-\Gamma^{p}_{nj}\,\Gamma^{i}_{mp}, \tag{5b}$$

$$\underset{3}{R}{}^{i}_{jmn}=\Gamma^{i}_{jm,n}-\Gamma^{i}_{nj,m}+\Gamma^{p}_{jm}\,\Gamma^{i}_{np}-\Gamma^{p}_{nj}\,\Gamma^{i}_{pm}+\Gamma^{p}_{nm}\left(\Gamma^{i}_{pj}-\Gamma^{i}_{jp}\right), \tag{5c}$$

$$\underset{3}{R}{}^{\alpha}_{\beta mn}=\left(\Gamma^{\alpha}_{\beta\mu,\nu}-\Gamma^{\alpha}_{\nu\beta,\mu}+\Gamma^{\pi}_{\beta\mu}\,\Gamma^{\alpha}_{\nu\pi}-\Gamma^{\pi}_{\nu\beta}\,\Gamma^{\alpha}_{\pi\mu}\right)t^{\mu}_m\,t^{\nu}_n+ \\ +2\,\Gamma^{\alpha}_{\underset{\vee}{\beta\mu}}\left(y^{\mu}_{,mn}-\Gamma^{p}_{nm}\,t^{\mu}_p\right), \tag{5d}$$

$$\underset{4}{R}{}^{\alpha}_{\beta mn}=\left(\Gamma^{\alpha}_{\beta\mu,\nu}-\Gamma^{\alpha}_{\nu\beta,\mu}+\Gamma^{\pi}_{\beta\mu}\,\Gamma^{\alpha}_{\nu\pi}-\Gamma^{\pi}_{\nu\beta}\,\Gamma^{\alpha}_{\pi\mu}\right)t^{\mu}_m\,t^{\nu}_n+ \\ +2\,\Gamma^{\alpha}_{\underset{\vee}{\beta\mu}}\left(y^{\mu}_{,mn}-\Gamma^{p}_{mn}\,t^{\mu}_p\right), \tag{5e}$$

and the notation $\underset{\vee}{mn}$ signifies the antisymmetrization on the indices m, n. The magnitudes $\underset{t}{R}{}^{\alpha}_{\beta\mu\nu}$, $\underset{t}{R}{}^{i}_{jmn}$ $(t=1, 2, 3)$ are tensors, and we call them curvature tensors of the space V_N, respectively the subspace V_M. of the 1 *st*, 2 *nd* and 3 *rd* kind. The magnitudes $\underset{3}{R}{}^{\alpha}_{\beta mn}$, $\underset{4}{R}{}^{\alpha}_{\beta mn}$ are tensors also, and we call them curvature tensors of the 3 *rd*, respectively 4 *th*, kind of the space V_N with respect to the subspace V_M.

In order to calculate the left sides in (4a-f), let us find first the covariant derivation of the kind λ, $\lambda \in \{1, 2, 3, 4\}$, of the formula (2a) on x^n:

$$t^{\alpha}_{i\underset{\theta}{|}m\underset{\lambda}{|}n} = \underset{\theta}{\Phi}{}^{p}_{im\underset{\lambda}{|}n}\, t^{\alpha}_{p} + \underset{\theta}{\Phi}{}^{p}_{im}\, t^{\alpha}_{p\underset{\lambda}{|}n} + \sum_{\rho} \underset{\theta}{\Omega}_{(\rho)im\underset{\lambda}{|}n} N^{\alpha}_{(\rho)} + \sum_{\rho} \underset{\theta}{\Omega}_{(\rho)im}\, N^{\alpha}_{(\rho)\underset{\lambda}{|}n}.$$

If we substitute $t^{\alpha}_{p\underset{\lambda}{|}n}$ in accordance with (2a) and $N^{\alpha}_{(\rho)\underset{\lambda}{|}n}$ in accordance with (2b) into the previous equation, we obtain

$$t^{\alpha}_{i\underset{\theta}{|}m\underset{\lambda}{|}n} = \underset{\theta}{\Phi}{}^{p}_{im\underset{\lambda}{|}n}\, t^{\alpha}_{p} + \underset{\theta}{\Phi}{}^{p}_{im}\big(\underset{\lambda}{\Phi}{}^{s}_{pn}\, t^{\alpha}_{s} + \sum_{\rho} \underset{\lambda}{\Omega}_{(\rho)pn} N^{\alpha}_{(\rho)}\big) + \sum_{\rho} \underset{\theta}{\Omega}_{(\rho)im\underset{\lambda}{|}n} N^{\alpha}_{(\rho)} + \sum_{\rho} \underset{\theta}{\Omega}_{(\rho)im}\big(-e_{(\rho)}\, g^{\underline{ps}}\, \underset{\lambda}{\Omega}_{(\rho)sn}\, t^{\alpha}_{p} + \sum_{\sigma} \underset{\lambda}{\Psi}_{(\sigma\rho)n} N^{\alpha}_{(\sigma)}\big),$$

i.e.

$$(6)\qquad t^{\alpha}_{i\underset{\theta}{|}m\underset{\lambda}{|}n} = \big[\underset{\theta}{\Phi}{}^{p}_{im\underset{\lambda}{|}n} + \underset{\theta}{\Phi}{}^{s}_{im}\underset{\lambda}{\Phi}{}^{p}_{sn} - \sum_{\rho} e_{(\rho)}\underset{\theta}{\Omega}_{(\rho)im}\,\underset{\lambda}{\Omega}_{(\rho)sn}\, g^{\underline{ps}}\big]\, t^{\alpha}_{p} + \sum_{\rho}\big[\underset{\lambda}{\Omega}_{(\rho)sn}\,\underset{\theta}{\Phi}{}^{s}_{im} + \underset{\theta}{\Omega}_{(\rho)im\underset{\lambda}{|}n} + \sum_{\sigma}\underset{\theta}{\Omega}_{(\sigma)im}\,\underset{\lambda}{\Psi}_{(\rho\sigma)n}\big] N^{\alpha}_{(\rho)}.$$

Hence

$$(7)\qquad \begin{aligned} t^{\alpha}_{i\underset{\theta}{|}m\underset{\lambda}{|}n} - t^{\alpha}_{i\underset{\lambda}{|}n\underset{\theta}{|}m} = {} & \big[\underset{\theta}{\Phi}{}^{p}_{im\underset{\lambda}{|}n} - \underset{\lambda}{\Phi}{}^{p}_{in\underset{\theta}{|}m} + \underset{\theta}{\Phi}{}^{s}_{im}\underset{\lambda}{\Phi}{}^{p}_{sn} - \\ & - \underset{\lambda}{\Phi}{}^{s}_{in}\,\underset{\theta}{\Phi}{}^{p}_{sm} - \sum_{\rho} e_{(\rho)}\, g^{\underline{ps}}\,(\underset{\lambda}{\Omega}_{(\rho)im}\underset{\theta}{\Omega}_{(\rho)sn} - \underset{\lambda}{\Omega}_{(\rho)in}\,\underset{\theta}{\Omega}_{(\rho)sm})\big]\, t^{\alpha}_{p} + \\ & + \sum_{\rho}\big[\underset{\theta}{\Omega}_{(\rho)im\underset{\lambda}{|}n} - \underset{\lambda}{\Omega}_{(\rho)in\underset{\theta}{|}m} + \underset{\theta}{\Phi}{}^{s}_{im}\underset{\lambda}{\Omega}_{(\rho)sn} - \underset{\lambda}{\Phi}{}^{s}_{in}\underset{\theta}{\Omega}_{(\rho)sm} + \\ & + \sum_{\sigma}\big(\underset{\theta}{\Omega}_{(\sigma)im}\,\underset{\lambda}{\Psi}_{(\rho\sigma)n} - \underset{\lambda}{\Omega}_{(\sigma)in}\,\underset{\theta}{\Psi}_{(\rho\sigma)m}\big)\big] N^{\alpha}_{(\rho)}. \end{aligned}$$

1.1. If in the previous equation we take $\theta=\lambda=1$ and in (4a) exchange $t^{\alpha}_{i\underset{1}{|}s}$ in accordance with (2a), by equalizing the right sides we get *the 1st integrability condition of the 1st derivational formula* (2a):

$$(8)\qquad \begin{aligned} & \underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\, t^{\mu}_{m}\, t^{\nu}_{n}\, t^{\pi}_{i} - \underset{1}{R}{}^{p}_{imn}\, t^{\alpha}_{p} - 2\Gamma^{s}_{mn}\big(\underset{1}{\Phi}{}^{p}_{is}\, t^{\alpha}_{p} + \sum_{\rho}\underset{1}{\Omega}_{(\rho)\,is} N^{\alpha}_{(\rho)}\big) = \\ & = \big[\underset{1}{\Phi}{}^{p}_{im\underset{1}{|}n} - \underset{1}{\Phi}{}^{p}_{in\underset{1}{|}m} + \underset{1}{\Phi}{}^{s}_{im}\,\underset{1}{\Phi}{}^{p}_{sn} - \underset{1}{\Phi}{}^{s}_{in}\underset{1}{\Phi}{}^{p}_{sm} - \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\rho} e_{(\rho)} g^{\underline{ps}} \big(\underset{1}{\Omega}_{(\rho)im} \underset{1}{\Omega}_{(\rho)sn} - \underset{1}{\Omega}_{(\rho)in} \underset{1}{\Omega}_{(\rho)sm}\big)\big] t^{\alpha}_{p} + \\
&+ \sum_{\rho} \big[\underset{1}{\Omega}_{(\rho)im \underset{1}{|} n} - \underset{1}{\Omega}_{(\rho)in \underset{1}{|} m} + \underset{1}{\Phi}^{s}_{im} \underset{1}{\Omega}_{(\rho)sn} - \Phi^{s}_{in} \Omega_{(\rho)sm} + \\
&+ \sum_{\sigma} \big(\underset{1}{\Omega}_{(\sigma)im} \underset{1}{\Psi}_{(\rho\sigma)n} - \underset{1}{\Omega}_{(\sigma)in} \underset{1}{\Psi}_{(\rho\sigma)m}\big)\big] N^{\alpha}_{(\rho)}.
\end{aligned}
$$

This equation for $e_{(\rho)}=1$ reduces to the equation (8) in [2].

If we multiply the previous equation by $a_{\underline{\alpha\beta}}\, t^{\beta}_{h}$, we get

$$
\begin{aligned}
&\underset{1}{R}_{\beta\pi\mu\nu}\, t^{\beta}_{h}\, t^{\pi}_{i}\, t^{\mu}_{m}\, t^{\nu}_{n} - \underset{1}{R}_{himn} - 2\, \underset{\vee}{\Gamma}^{s}_{mn} \underset{1}{\Phi}_{his} = \\
&= \underset{1}{\Phi}_{him \underset{1}{|} n} - \underset{1}{\Phi}_{hin \underset{1}{|} m} + \underset{1}{\Phi}^{s}_{im} \Phi_{hsn} - \underset{1}{\Phi}^{s}_{in} \underset{1}{\Phi}_{hsm} + \\
&+ \sum_{\rho} e_{(\rho)} \big(\underset{1}{\Omega}_{(\rho)hm} \underset{1}{\Omega}_{(\rho)in} - \underset{1}{\Omega}_{(\rho)hn} \underset{1}{\Omega}_{(\rho)im}\big),
\end{aligned}
\tag{9}
$$

which is *the Gauss equation of the 1st kind.* Here

$$
\underset{1}{R}_{\beta\pi\mu\nu} = a_{\underline{\alpha\beta}} \underset{1}{R}^{\alpha}_{\pi\mu\nu}, \quad \underset{1}{R}_{himn} = g_{\underline{ph}} \underset{1}{R}^{p}_{imn}.
$$

By multiplying (8) by $a_{\underline{\alpha\beta}}\, N^{\beta}_{(\tau)}$ one obtains

$$
\begin{aligned}
&\underset{1}{R}_{\beta\pi\mu\nu}\, N^{\beta}_{(\tau)}\, t^{\pi}_{i}\, t^{\mu}_{m}\, t^{\nu}_{n} - 2\, \underset{\vee}{\Gamma}^{s}_{mn}\, e_{(\tau)} \underset{1}{\Omega}_{(\tau)is} = \\
&= e_{(\tau)} \big[\underset{1}{\Omega}_{(\tau)im \underset{1}{|} n} - \underset{1}{\Omega}_{(\tau)in \underset{1}{|} m} + \underset{1}{\Phi}^{s}_{im} \underset{1}{\Omega}_{(\tau)sn} - \underset{1}{\Phi}^{s}_{in} \underset{1}{\Omega}_{(\tau)sm} + \\
&+ \sum_{\sigma} \big(\underset{1}{\Omega}_{(\sigma)im} \underset{1}{\Psi}_{(\tau\sigma)n} - \underset{1}{\Omega}_{(\sigma)in} \underset{1}{\Psi}_{(\tau\sigma)m}\big)\big].
\end{aligned}
\tag{10}
$$

This is *the 1st Codazzi equation of the 1st kind.*

1.2. *The 2nd integrability condition of the 1st derivational formula* we obtain if we put in (7) $\theta=\lambda=2$ and in (4b) exchange $t^{\alpha}_{i\underset{2}{|}s}$ in accordance with (2a), and then equalize the right sides:

$$
\begin{aligned}
&\underset{2}{R}^{\alpha}_{\pi\mu\nu}\, t^{\mu}_{m}\, t^{\nu}_{n}\, t^{\pi}_{i} - \underset{2}{R}^{p}_{imn}\, t^{\alpha}_{p} + 2\, \underset{\vee}{\Gamma}^{s}_{mn} \big(\underset{2}{\Phi}^{p}_{is}\, t^{\alpha}_{p} + \sum_{\rho} \underset{2}{\Omega}_{(\rho)is}\, N^{\alpha}_{(\rho)}\big) = \\
&= \big[\underset{2}{\Phi}^{p}_{im \underset{2}{|} n} - \underset{2}{\Phi}^{p}_{in \underset{2}{|} m} + \underset{2}{\Phi}^{s}_{im} \underset{2}{\Phi}^{p}_{sn} - \underset{2}{\Phi}^{s}_{in} \underset{2}{\Phi}^{p}_{sm} - \\
&- \sum_{\rho} e_{(\rho)}\, g^{\underline{ps}} \big(\underset{2}{\Omega}_{(\rho)im} \underset{2}{\Omega}_{(\rho)sn} - \underset{2}{\Omega}_{(\rho)in} \underset{2}{\Omega}_{(\rho)sm}\big)\big] t^{\alpha}_{p} + \\
&+ \sum_{\rho} \big[\underset{2}{\Omega}_{(\rho)im \underset{2}{|} n} - \underset{2}{\Omega}_{(\rho)in \underset{2}{|} m} + \underset{2}{\Phi}^{s}_{im} \underset{2}{\Omega}_{(\rho)sn} \underset{2}{\Phi}^{s}_{in} \underset{2}{\Omega}_{(\rho)sm} + \\
&+ \sum_{\sigma} \big(\underset{2}{\Omega}_{(\sigma)im} \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\rho)in} \underset{2}{\Psi}_{(\rho\sigma)m}\big)\big] N^{\alpha}_{(\rho)}.
\end{aligned}
\tag{11}
$$

Multiplying this equation by $a_{\underline{\alpha\beta}}\, t^\beta_h$, we obtain *the Gauss equation of the 2nd kind*:

$$
\begin{aligned}
&\underset{2}{R}_{\beta\pi\mu\nu}\, t^\beta_h\, t^\pi_i\, t^\mu_m\, t^\nu_n - \underset{2}{R}_{himn} + 2\,\underset{\vee}{\Gamma}{}^s_{mn}\, \underset{2}{\Phi}_{his} = \\
&= \underset{2}{\Phi}_{him\underset{2}{|}n} - \underset{2}{\Phi}_{hin\underset{2}{|}m} + \underset{2}{\Phi}{}^s_{im}\, \underset{2}{\Phi}_{hsn} - \underset{2}{\Phi}{}^s_{in}\, \underset{2}{\Phi}_{hsm} + \\
&+ \sum_\rho e_{(\rho)} \big(\underset{2}{\Omega}_{(\rho)hm}\, \underset{2}{\Omega}_{(\rho)in} - \underset{2}{\Omega}_{(\rho)hn}\, \underset{2}{\Omega}_{(\rho)im} \big),
\end{aligned}
\tag{12}
$$

where

$$
\underset{2}{R}_{\beta\pi\mu\nu} = a_{\underline{\alpha\beta}}\, \underset{2}{R}{}^\alpha_{\pi\mu\nu}\,, \quad \underset{2}{R}_{himn} = g_{\underline{ph}}\, \underset{2}{R}{}^p_{imn}\,.
$$

Multiplying the equation (11) by $a_{\underline{\alpha\beta}}\, N^\beta_{(\tau)}$, we get

$$
\begin{aligned}
&\underset{2}{R}_{\beta\pi\mu\nu}\, N^\beta_{(\tau)}\, t^\pi_i\, t^\mu_m\, t^\nu_n + 2\,\underset{\vee}{\Gamma}{}^s_{mn}\, e_{(\tau)}\, \underset{2}{\Omega}_{(\tau)is} = \\
&= e_{(\tau)} \big[\underset{2}{\Omega}_{(\tau)im\underset{2}{|}n} - \underset{2}{\Omega}_{(\tau)in\underset{2}{|}m} + \underset{2}{\Phi}{}^s_{im}\, \underset{2}{\Omega}_{(\tau)sn} - \underset{2}{\Phi}{}^s_{in}\, \underset{2}{\Omega}_{(\tau)sm} + \\
&+ \sum_\sigma \big(\underset{2}{\Omega}_{(\sigma)im}\, \underset{2}{\Psi}_{(\tau\sigma)n} - \underset{2}{\Omega}_{(\sigma)in}\, \underset{2}{\Psi}_{(\tau\sigma)m} \big) \big],
\end{aligned}
\tag{13}
$$

which is *the 1st Codazzi equation of the 2nd kind.*

1.3. If in (7) we put $\theta = 1$, $\lambda = 2$ and then equalize the right sides of this equation and of the equation (4c), we get *the 3rd integrability condition of the 1st derivational formula*:

$$
\begin{aligned}
&\underset{3}{R}{}^\alpha_{\pi mn}\, t^\pi_i - \underset{3}{R}{}^p_{imn}\, t^\alpha_p = \\
&= \big[\underset{1}{\Phi}{}^p_{im\underset{2}{|}n} - \underset{2}{\Phi}{}^p_{in\underset{1}{|}m} + \underset{1}{\Phi}{}^s_{im}\, \underset{2}{\Phi}{}^p_{sn} - \underset{2}{\Phi}{}^s_{in}\, \underset{1}{\Phi}{}^p_{sm} - \\
&- \sum_\rho e_{(\rho)}\, g^{\underline{ps}} \big(\underset{1}{\Omega}_{(\rho)im}\, \underset{2}{\Omega}_{(\rho)sn} - \underset{2}{\Omega}_{(\rho)in}\, \underset{1}{\Omega}_{(\rho)sm} \big) \big]\, t^\alpha_p + \\
&+ \sum_\rho \big[\underset{1}{\Omega}_{(\rho)im\underset{2}{|}n} - \underset{2}{\Omega}_{(\rho)in\underset{1}{|}m} + \underset{1}{\Phi}{}^s_{im}\, \underset{2}{\Omega}_{(\rho)sn} - \underset{2}{\Phi}{}^s_{in}\, \underset{1}{\Omega}_{(\rho)sm} + \\
&+ \sum_\sigma \big(\underset{1}{\Omega}_{(\sigma)im}\, \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\sigma)in}\, \underset{1}{\Psi}_{(\rho\sigma)m} \big) \big]\, N^\alpha_{(\rho)}\,.
\end{aligned}
\tag{14}
$$

If me multiply this equation by $a_{\underline{\alpha\beta}}\, t^\beta_h$, we obtain *the Gauss equation of the 3rd kind*:

$$
\begin{aligned}
&\underset{3}{R}_{\beta\pi mn}\, t^\beta_h\, t^\pi_i - \underset{3}{R}_{himn} = \\
&= \underset{1}{\Phi}_{him\underset{2}{|}n} - \underset{2}{\Phi}_{hin\underset{1}{|}m} + \underset{1}{\Phi}{}^s_{im}\, \underset{2}{\Phi}_{hsn} - \underset{2}{\Phi}{}^s_{in}\, \underset{1}{\Phi}_{hsm} + \\
&+ \sum_\rho e_{(\rho)} \big(\underset{1}{\Omega}_{(\rho)hm}\, \underset{2}{\Omega}_{(\rho)in} - \underset{2}{\Omega}_{(\rho)hn}\, \underset{1}{\Omega}_{(\rho)im} \big),
\end{aligned}
\tag{15}
$$

where

$$\underset{3}{R}_{\beta\pi mn}=\underline{a_{\alpha\beta}}\,\underset{3}{R}{}^{\alpha}_{\pi mn},\quad \underset{3}{R}_{himn}=\underline{g_{ph}}\,\underset{3}{R}{}^{p}_{imn}\,.$$

Multiplying (14) by $\underline{a_{\alpha\beta}}N^{\beta}_{(\tau)}$, we get *the 1st Codazzi equation of the 3rd kind:*

$$
\begin{aligned}
(16)\qquad \underset{3}{R}_{\beta\pi mn}N^{\beta}_{(\tau)}\,t^{\pi}_{i} &= e_{(\tau)}\big[\underset{1}{\Omega}_{(\tau)im\underset{2}{|}n}-\underset{2}{\Omega}_{(\tau)in\underset{1}{|}m}+\\
&+\underset{1}{\Phi}{}^{s}_{im}\,\underset{2}{\Omega}_{(\tau)sn}-\underset{2}{\Phi}{}^{s}_{in}\,\underset{1}{\Omega}_{(\tau)sm}+\\
&+\sum_{\sigma}\big(\underset{1}{\Omega}_{(\sigma)im}\,\underset{2}{\Psi}_{(\tau\sigma)n}-\underset{2}{\Omega}_{(\sigma)in}\,\underset{1}{\Psi}_{(\tau\sigma)m}\big)\big].
\end{aligned}
$$

1.4. If in (7) we take $\theta=\lambda=3$, in (4d) exchange $t^{\alpha}_{i\underset{3}{|}s}$ in accordance with (2a), and then equalize the right sides of these equations, we obtain *the 4 th integrability condition of the 1st derivational formula*

$$
\begin{aligned}
&\underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\,t^{\pi}_{i}\,t^{\mu}_{m}\,t^{\nu}_{n}-\underset{2}{R}{}^{p}_{imn}\,t^{\alpha}_{p}+2\,\Gamma^{s}_{\underset{\vee}{mn}}\big(\underset{3}{\Phi}{}^{p}_{is}\,t^{\alpha}_{p}+\sum_{\rho}\underset{3}{\Omega}_{(\rho)is}N^{\alpha}_{(\rho)}\big)=\\
&=\big[\underset{3}{\Phi}{}^{p}_{im\underset{3}{|}n}-\underset{3}{\Phi}{}^{p}_{in\underset{3}{|}m}+\underset{3}{\Phi}{}^{s}_{im}\,\underset{3}{\Phi}{}^{p}_{sn}-\underset{3}{\Phi}{}^{s}_{in}\,\underset{3}{\Phi}{}^{p}_{sm}-\\
(17)\qquad &-\sum_{\rho}e_{(\rho)}\,\underline{g^{ps}}\big(\underset{3}{\Omega}_{(\rho)im}\underset{3}{\Omega}_{(\rho)sn}-\underset{3}{\Omega}_{(\rho)in}\,\underset{3}{\Omega}_{(\rho)sm}\big)\big]\,t^{\alpha}_{p}+\\
&+\sum_{\rho}\big[\underset{3}{\Omega}_{(\rho)im\underset{3}{|}n}-\underset{3}{\Omega}_{(\rho)in\underset{3}{|}m}+\underset{3}{\Phi}{}^{s}_{im}\underset{3}{\Omega}_{(\rho)sn}-\underset{3}{\Phi}{}^{s}_{in}\,\underset{3}{\Omega}_{(\rho)sm}+\\
&+\sum_{\sigma}\big(\underset{3}{\Omega}_{(\rho)im}\,\underset{3}{\Psi}_{(\rho\sigma)n}-\underset{3}{\Omega}_{(\sigma)in}\,\underset{3}{\Psi}_{(\rho\sigma)m}\big)\big]\,N^{\alpha}_{(\rho)}\,.
\end{aligned}
$$

Based on (3'a), (3''a) and taking into account that for a tensor having only covariant indices the third kind of differentiation reduces to the second, while the forth kind reduces to the first, the equation (17) becomes

$$
\begin{aligned}
&\underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\,t^{\pi}_{i}\,t^{\mu}_{m}\,t^{\nu}_{n}-\underset{2}{R}{}^{p}_{imn}\,t^{\alpha}_{p}+2\,\Gamma^{s}_{\underset{\vee}{mn}}\big(\underset{3}{\Phi}{}^{p}_{is}\,t^{\alpha}_{p}+\sum_{\rho}\underset{1}{\Omega}_{(\rho)is}\,N^{\alpha}_{(\rho)}\big)=\\
&=\big[\underset{3}{\Phi}{}^{p}_{im\underset{3}{|}n}-\underset{3}{\Phi}{}^{p}_{in\underset{3}{|}m}+\underset{3}{\Phi}{}^{s}_{im}\,\underset{3}{\Phi}{}^{p}_{sn}-\underset{3}{\Phi}{}^{s}_{in}\,\underset{3}{\Phi}{}^{p}_{sm}-\\
(17')\qquad &-\sum_{\rho}e_{(\rho)}\,\underline{g^{ps}}\big(\underset{1}{\Omega}_{(\rho)im}\,\underset{1}{\Omega}_{(\rho)sn}-\underset{1}{\Omega}_{(\rho)in}\underset{1}{\Omega}_{(\rho)sm}\big)\big]\,t^{\alpha}_{p}+\\
&+\sum_{\rho}\big[\underset{1}{\Omega}_{(\rho)im\underset{2}{|}n}-\underset{1}{\Omega}_{(\rho)in\underset{2}{|}m}+\underset{3}{\Phi}{}^{s}_{im}\underset{1}{\Omega}_{(\rho)sn}-\underset{3}{\Phi}{}^{s}_{in}\underset{1}{\Omega}_{(\rho)sm}+\\
&+\sum_{\sigma}\big(\underset{1}{\Omega}_{(\sigma)im}\,\underset{1}{\Psi}_{(\rho\sigma)n}-\underset{1}{\Omega}_{(\sigma)in}\underset{1}{\Psi}_{(\rho\sigma)m}\big)\big]N^{\alpha}_{(\rho)}.
\end{aligned}
$$

Multiplying the previous equation by $\underline{a}_{\alpha\beta}\, t^{\beta}_{h}$, we obtain *the Gauss equation of the 4 th kind*

$$\begin{aligned}&\underset{1}{R}_{\beta\pi\mu\nu}\, t^{\beta}_{h}\, t^{\pi}_{i}\, t^{\mu}_{m}\, t^{\nu}_{n} - \underset{2}{R}_{himn} + 2\,\underset{\vee}{\Gamma}{}^{s}_{mn}\,\underset{3}{\Phi}_{his} =\\ &= \underline{g}_{ph}\big(\underset{3}{\Phi}{}^{p}_{im\underset{3}{|}n} - \underset{3}{\Phi}{}^{p}_{in\underset{3}{|}m}\big) + \underset{3}{\Phi}{}^{s}_{im}\,\underset{3}{\Phi}_{hsn} - \underset{3}{\Phi}{}^{s}_{in}\,\underset{3}{\Phi}_{hsm} +\\ &+ \sum_{\rho} e_{(\rho)}\big(\underset{1}{\Omega}_{(\rho)hm}\,\underset{1}{\Omega}_{(\rho)in} - \underset{1}{\Omega}_{(\rho)hn}\,\underset{1}{\Omega}_{(\rho)im}\big).\end{aligned} \tag{18}$$

We remark that

$$\underline{g}_{ph}\,\underset{3}{\Phi}{}^{p}_{im\underset{3}{|}n} \neq \underset{3}{\Phi}_{him\underset{3}{|}n}\,.$$

If we multiply the equation (17′) by $\underline{a}_{\alpha\beta}\, N^{\beta}_{(\tau)}$, we get *the 1st Codazzi equation of the 4 th kind*

$$\begin{aligned}&\underset{1}{R}_{\beta\pi\mu\nu}\, N^{\beta}_{(\tau)}\, t^{\pi}_{i}\, t^{\mu}_{m}\, t^{\nu}_{n} + 2\,\underset{\vee}{\Gamma}{}^{s}_{mn}\, e_{(\tau)}\,\underset{1}{\Omega}_{(\tau)is} =\\ &= e_{(\tau)}\big[\underset{1}{\Omega}_{(\tau)im\underset{2}{|}n} - \underset{1}{\Omega}_{(\tau)in\underset{2}{|}m} + \underset{3}{\Phi}{}^{s}_{im}\,\underset{1}{\Omega}_{(\tau)sn} - \underset{3}{\Phi}{}^{s}_{in}\,\underset{1}{\Omega}_{(\tau)sm} +\\ &+ \sum_{\sigma}\big(\underset{1}{\Omega}_{(\sigma)im}\,\underset{1}{\Psi}_{(\tau\sigma)n} - \underset{1}{\Omega}_{(\sigma)in}\,\underset{1}{\Psi}_{(\tau\sigma)m}\big)\big].\end{aligned} \tag{19}$$

We shall prove that the equation (19) is equivalent to the equation (10). Namely, since

$$\begin{aligned}&\underset{1}{\Omega}_{(\tau)im\underset{1}{|}n} - \underset{1}{\Omega}_{(\tau)in\underset{1}{|}m} =\\ &= \underset{1}{\Omega}_{(\tau)im,n} - \underset{1}{\Omega}_{(\tau)in,m} + \Gamma^{r}_{im}\,\underset{1}{\Omega}_{(\tau)rn} - \Gamma^{r}_{in}\,\underset{1}{\Omega}_{(\tau)rm} - 2\,\underset{\vee}{\Gamma}{}^{r}_{mn}\,\underset{1}{\Omega}_{(\tau)ir}\,,\end{aligned} \tag{19a}$$

$$\begin{aligned}&\underset{1}{\Omega}_{(\tau)im\underset{2}{|}n} - \underset{1}{\Omega}_{(\tau)in\underset{2}{|}m} =\\ &= \underset{1}{\Omega}_{(\tau)im,n} - \underset{1}{\Omega}_{(\tau)in,m} + \Gamma^{r}_{mi}\,\underset{1}{\Omega}_{(\tau)rn} - \Gamma^{r}_{ni}\,\underset{1}{\Omega}_{(\tau)rm} + 2\,\underset{\vee}{\Gamma}{}^{r}_{mn}\,\underset{1}{\Omega}_{(\tau)ir}\,,\end{aligned} \tag{19b}$$

and, based on (48′b) in [5],

$$\underset{3}{\Phi}{}^{s}_{im} = \underset{1}{\Phi}{}^{s}_{im} + 2\,\underset{\vee}{\Gamma}{}^{s}_{im}\,,$$

the equations (10) and (19) reduce to the same equation:

$$\begin{aligned}&\underset{1}{R}_{\beta\pi\mu\nu}\, N^{\beta}_{(\tau)}\, t^{\pi}_{i}\, t^{\mu}_{m}\, t^{\nu}_{n} = e_{(\tau)}\big[\underset{1}{\Omega}_{(\tau)im,n} - \underset{1}{\Omega}_{(\tau)in,m} +\\ &+ \Gamma^{r}_{im}\,\underset{1}{\Omega}_{(\tau)rn} - \Gamma^{r}_{in}\,\underset{1}{\Omega}_{(\tau)rm} + \underset{1}{\Phi}{}^{s}_{im}\,\underset{1}{\Omega}_{(\tau)sn} - \underset{1}{\Phi}{}^{s}_{in}\,\underset{1}{\Omega}_{(\tau)sm} +\\ &+ \sum_{\sigma}\big(\underset{1}{\Omega}_{(\sigma)im}\,\underset{1}{\Psi}_{(\tau\sigma)n} - \underset{1}{\Omega}_{(\sigma)in}\,\underset{1}{\Psi}_{(\tau\sigma)m}\big)\big].\end{aligned} \tag{19′}$$

This means that the equations (10), (19) and (19′) are *three forms of the 1st Codazzi equation of the 1 st kind (10).*

1.5. Puting into (7) $\theta=\lambda=4$, and substituting in (4 *e*) $t^{\alpha}_{i\underset{4}{|}s}$ in accordance with (2*a*) and equalizing the right sides of the obtained equations, we obtain *the 5 th integrability condition of the 1st derivational formula*

$$
\begin{aligned}
&\underset{2}{R}^{\alpha}_{\pi\mu\nu}\, t^{\mu}_{m}\, t^{\nu}_{n}\, t^{\pi}_{i} - \underset{1}{R}^{p}_{imn}\, t^{\alpha}_{p} - 2\,\underset{\vee}{\Gamma}^{s}_{mn}\Big(\underset{4}{\Phi}^{p}_{is}\, t^{\alpha}_{p} + \sum_{\rho} \underset{4}{\Omega}_{(\rho)\,is}\, N^{\alpha}_{(\rho)}\Big) = \\
&= \Big[\underset{4}{\Phi}^{p}_{im\underset{4}{|}n} - \underset{4}{\Phi}^{p}_{in\underset{4}{|}m} + \underset{4}{\Phi}^{s}_{im}\underset{4}{\Phi}^{p}_{sn} - \underset{4}{\Phi}^{s}_{in}\underset{4}{\Phi}^{p}_{sm} - \\
(20)\qquad &- \sum_{\rho} e_{(\rho)}\, g^{\underline{ps}}\big(\underset{4}{\Omega}_{(\rho)im}\,\underset{4}{\Omega}_{(\rho)sn} - \underset{4}{\Omega}_{(\rho)in}\,\underset{4}{\Omega}_{(\rho)sm}\big)\Big]\, t^{\alpha}_{p} + \\
&+ \sum_{\rho}\Big[\underset{4}{\Omega}_{(\rho)im\underset{4}{|}n} - \underset{4}{\Omega}_{(\rho)in\underset{4}{|}m} + \underset{4}{\Phi}^{s}_{im}\underset{4}{\Omega}_{(\rho)sn} - \underset{4}{\Phi}^{s}_{in}\underset{4}{\Omega}_{(\rho)sm} + \\
&+ \sum_{\sigma}\big(\underset{4}{\Omega}_{(\sigma)im}\,\underset{4}{\Psi}_{(\rho\sigma)n} - \underset{4}{\Omega}_{(\sigma)in}\,\underset{4}{\Psi}_{(\rho\sigma)m}\big)\Big] N^{\alpha}_{(\rho)}\,.
\end{aligned}
$$

Taking into account (3′b), (3″b) and multiplying the equation (20) by $a_{\underline{\alpha\beta}}\, t^{\beta}_{h}$ we obtain *the Gauss equation of the 5 th kind*

$$
\begin{aligned}
&\underset{2}{R}_{\beta\pi\mu\nu}\, t^{\beta}_{h}\, t^{\pi}_{i}\, t^{\mu}_{m}\, t^{\nu}_{n} - \underset{1}{R}_{himn} - 2\,\underset{\vee}{\Gamma}^{s}_{mn}\underset{4}{\Phi}_{his} = \\
(21)\qquad &= g_{\underline{ph}}\big(\underset{4}{\Phi}^{p}_{im\underset{4}{|}n} - \underset{4}{\Phi}^{p}_{in\underset{4}{|}m}\big) + \underset{4}{\Phi}^{s}_{im}\underset{4}{\Phi}_{hsn} - \underset{4}{\Phi}^{s}_{in}\underset{4}{\Phi}_{hsm} + \\
&+ \sum_{\rho} e_{(\rho)}\, g^{\underline{ps}}\big(\underset{2}{\Omega}_{(\rho)hm}\underset{2}{\Omega}_{(\rho)in} - \underset{2}{\Omega}_{(\rho)hn}\underset{2}{\Omega}_{(\rho)im}\big)\,.
\end{aligned}
$$

Multiplying the equation (20) by $a_{\underline{\alpha\beta}}\, N^{\beta}_{(\tau)}$, we get *the 1st Codazzi equation of the 5 th kind.*

$$
\begin{aligned}
&\underset{2}{R}_{\beta\pi\mu\nu}\, N^{\beta}_{(\tau)}\, t^{\pi}_{i}\, t^{\mu}_{m}\, t^{\nu}_{n} - 2\,\underset{\vee}{\Gamma}^{s}_{mn}\, e_{(\tau)}\underset{2}{\Omega}_{(\tau)is} = \\
(22)\qquad &= e_{(\tau)}\Big[\underset{2}{\Omega}_{(\tau)im\underset{1}{|}n} - \underset{2}{\Omega}_{(\tau)in\underset{1}{|}m} + \underset{4}{\Phi}^{s}_{im}\underset{2}{\Omega}_{(\tau)sn} - \underset{4}{\Phi}^{s}_{in}\underset{2}{\Omega}_{(\tau)sm} + \\
&+ \sum_{\sigma}\big(\underset{2}{\Omega}_{(\sigma)im}\underset{2}{\Psi}_{(\tau\sigma)n} - \underset{2}{\Omega}_{(\sigma)in}\underset{2}{\Psi}_{(\tau\sigma)m}\big)\Big].
\end{aligned}
$$

We have to prove that the equation (22) is equivalent to the equation (13). Since

$$
\begin{aligned}
&\underset{2}{\Omega}_{(\tau)im\underset{2}{|}n} - \underset{2}{\Omega}_{(\tau)in\underset{2}{|}m} = \\
(22a)\qquad &= \underset{2}{\Omega}_{(\tau)im,n} - \underset{2}{\Omega}_{(\tau)in,m} + \Gamma^{r}_{mi}\underset{2}{\Omega}_{(\tau)rn} - \Gamma^{r}_{ni}\underset{2}{\Omega}_{(\tau)rm} + 2\,\underset{\vee}{\Gamma}^{r}_{mn}\underset{2}{\Omega}_{(\tau)ir}\,,
\end{aligned}
$$

$$\underset{2}{\Omega}_{(\tau)im\underset{1}{|}n} - \underset{2}{\Omega}_{(\tau)in\underset{1}{|}m} =$$

(22b)

$$= \underset{2}{\Omega}_{(\tau)im,n} - \underset{2}{\Omega}_{(\tau)in,m} + \Gamma^r_{im}\underset{2}{\Omega}_{(\tau)rn} - \Gamma^r_{in}\underset{2}{\Omega}_{(\tau)rm} - 2\underset{\vee}{\Gamma}{}^r_{mn}\underset{2}{\Omega}_{(\tau)ir},$$

and based on (48′c, a) in [5]

$$\underset{4}{\Phi}{}^s_{im} = \underset{2}{\Phi}{}^s_{im} - 2\underset{\vee}{\Gamma}{}^s_{im},$$

the equations (13) and (22) reduce to the same equation

$$\underset{2}{R}_{\beta\pi\mu\nu} N^\beta_{(\tau)} t^\pi_i t^\mu_m t^\nu_n = e_{(\tau)}\big[\underset{2}{\Omega}_{(\tau)im,n} - \underset{2}{\Omega}_{(\tau)in,m} +$$

(22′)

$$+ \Gamma^r_{mi}\underset{2}{\Omega}_{(\tau)rn} - \Gamma^r_{ni}\underset{2}{\Omega}_{(\tau)rm} + \underset{2}{\Phi}{}^s_{im}\underset{2}{\Omega}_{(\tau)sn} - \underset{2}{\Phi}{}^s_{in}\underset{2}{\Omega}_{(\tau)sm} +$$

$$+ \sum_\sigma\big(\underset{2}{\Omega}_{(\sigma)im}\underset{2}{\Psi}{}_{(\tau\sigma\, n} - \underset{2}{\Omega}_{(\sigma)in}\underset{2}{\Psi}{}_{(\tau\sigma)m}\big)\big].$$

So, the equations (13), (22) and (22′) are *three forms of the 1st Codazzi equation of the 2nd kind (13)*.

1.6. In order to obtain *the 6 th integrability condition of the 1st derivational formula*, we shall put in (7) $\theta = 3$, $\lambda = 4$ and then we shall equalize the right side of the obtained equation with the right side in (4f):

$$\underset{4}{R}{}^\alpha_{\pi mn} t^\pi_i + \underset{3}{R}{}^p_{inm} t^\alpha_p =$$

(23)

$$= \big[\underset{3}{\Phi}{}^p_{im\underset{4}{|}n} - \underset{4}{\Phi}{}^p_{in\underset{3}{|}m} + \underset{3}{\Phi}{}^s_{im}\underset{4}{\Phi}{}^p_{sn} - \underset{4}{\Phi}{}^s_{in}\underset{3}{\Phi}{}^p_{sm} -$$

$$- \sum_\rho e_{(\rho)} g^{\underline{ps}}\big(\underset{3}{\Omega}_{(\rho)im}\underset{4}{\Omega}_{(\rho)sn} - \underset{4}{\Omega}_{(\rho)in}\underset{3}{\Omega}_{(\rho)sm}\big)\big] t^\alpha_p +$$

$$+ \sum_\rho\big[\underset{3}{\Omega}_{(\rho)im\underset{4}{|}n} - \underset{4}{\Omega}_{(\rho)in\underset{3}{|}m} + \underset{3}{\Phi}{}^s_{im}\underset{4}{\Omega}_{(\rho)sn} - \underset{4}{\Phi}{}^s_{in}\underset{3}{\Omega}_{(\rho)sm} +$$

$$+ \sum_\sigma\big(\underset{3}{\Omega}_{(\sigma)im}\underset{4}{\Psi}{}_{(\rho\sigma)n} - \underset{4}{\Omega}_{(\sigma)in}\underset{3}{\Psi}{}_{(\rho\sigma)m}\big)\big] N^\alpha_{(\rho)}.$$

Multiplying this equation by $a_{\alpha\beta}\, t^\beta_h$ and taking into account (3′), (3″), we obtain *the Gauss equation of the 6th kind*

$$\underset{4}{R}_{\beta\pi mn} t^\beta_h t^\pi_i + \underset{3}{R}_{hinm} =$$

(24)

$$= \underline{g_{ph}}\big(\underset{3}{\Phi}{}^p_{im\underset{4}{|}n} - \underset{4}{\Phi}{}^p_{in\underset{3}{|}m}\big) + \underset{3}{\Phi}{}^s_{im}\underset{4}{\Phi}_{hsn} - \underset{4}{\Phi}{}^s_{in}\underset{3}{\Phi}_{hsm} +$$

$$+ \sum_\rho e_{(\rho)} g^{\underline{ps}}\big(\underset{1}{\Omega}_{(\rho)hm}\underset{2}{\Omega}_{(\rho)in} - \underset{2}{\Omega}_{(\rho)hn}\underset{1}{\Omega}_{(\rho)im}\big),$$

where

$$\underset{4}{R}_{\beta\pi mn} = a_{\underline{\alpha\beta}} \underset{4}{R}^{\alpha}_{\pi mn}.$$

If we multiply the equation (23) by $a_{\underline{\alpha\beta}} N^{\beta}_{(\tau)}$, we obtain the *1st Codazzi equation of the 6 th kind*

$$(25)\quad \begin{aligned} \underset{4}{R}_{\beta\pi mn} N^{\beta}_{(\tau)} t^{\pi}_{i} = e_{(\tau)} \big[& \underset{1}{\Omega}_{(\tau)im\underset{1}{|}n} - \underset{2}{\Omega}_{(\tau)in\underset{2}{|}m} + \underset{3}{\Phi}^{s}_{im} \underset{2}{\Omega}_{(\tau)sn} - \\ & - \underset{4}{\Phi}^{s}_{in} \underset{1}{\Omega}_{(\tau)sm} + \sum_{\sigma} \big(\underset{1}{\Omega}_{(\sigma)im} \underset{2}{\Psi}_{(\tau\sigma)n} - \underset{2}{\Omega}_{(\sigma)in} \underset{1}{\Psi}_{(\tau\sigma)m} \big) \big]. \end{aligned}$$

2. Integrability conditions of the second derivational formula

2.0. Applying the Ricci-type identities (7), (11), (56) from [3] and (12), (13), (46) from [4] to normals $N^{\alpha}_{(\sigma)}$ we get

$$(26a)\quad N^{\alpha}_{(\sigma)\underset{1}{|}mn} - N^{\alpha}_{(\sigma)\underset{1}{|}nm} = \underset{1}{R}^{\alpha}_{\pi\mu\nu} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} - 2\,\Gamma^{q}_{\underset{\vee}{mn}} N^{\alpha}_{(\sigma)\underset{1}{|}q},$$

$$(26b)\quad N^{\alpha}_{(\sigma)\underset{2}{|}mn} - N^{\alpha}_{(\sigma)\underset{2}{|}nm} = \underset{2}{R}^{\alpha}_{\pi\mu\nu} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} + 2\,\Gamma^{q}_{\underset{\vee}{mn}} N^{\alpha}_{(\sigma)\underset{2}{|}q},$$

$$(26c)\quad N^{\alpha}_{(\sigma)\underset{1}{|}m\underset{2}{|}n} - N^{\alpha}_{(\sigma)\underset{2}{|}n\underset{1}{|}m} = \underset{3}{R}^{\alpha}_{\pi mn} N^{\pi}_{(\sigma)},$$

$$(26d)\quad N^{\alpha}_{(\sigma)\underset{3}{|}mn} - N^{\alpha}_{(\sigma)\underset{3}{|}nm} = \underset{1}{R}^{\alpha}_{\pi\mu\nu} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} + 2\,\Gamma^{q}_{\underset{\vee}{mn}} N^{\alpha}_{(\sigma)\underset{3}{|}q},$$

$$(26e)\quad N^{\alpha}_{(\sigma)\underset{4}{|}mn} - N^{\alpha}_{(\sigma)\underset{4}{|}nm} = \underset{4}{R}^{\alpha}_{\pi\mu\nu} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} - 2\,\Gamma^{q}_{\underset{\vee}{mn}} N^{\alpha}_{(\sigma)\underset{4}{|}q},$$

$$(26f)\quad N^{\alpha}_{(\sigma)\underset{3}{|}m\underset{4}{|}n} - N^{\alpha}_{(\sigma)\underset{4}{|}n\underset{3}{|}m} = \underset{4}{R}^{\alpha}_{\pi mn} N^{\pi}_{(\sigma)}.$$

Let us now find the covariant derivation of the kind λ, $\lambda \in \{1, 2, 3, 4\}$ of the formula (2b) on x^n:

$$\begin{aligned} N^{\alpha}_{(\sigma)\underset{\theta}{|}m\underset{\lambda}{|}n} = & - e_{(\sigma)}\, g^{\underline{ps}} \underset{\theta}{\Omega}_{(\sigma)sm\underset{\lambda}{|}n} t^{\alpha}_{p} - e_{(\sigma)}\, g^{\underline{ps}} \underset{\theta}{\Omega}_{(\sigma)sm} t^{\alpha}_{p\underset{\lambda}{|}n} + \\ & + \sum_{\rho} \underset{\theta}{\Psi}_{(\rho\sigma)m\underset{\lambda}{|}n} N^{\alpha}_{(\rho)} + \sum_{\rho} \underset{\theta}{\Psi}_{(\rho\sigma)m} N^{\alpha}_{(\rho)\underset{\lambda}{|}n}. \end{aligned}$$

Putting $t^{\alpha}_{p\underset{\lambda}{|}n}$ and $N^{\alpha}_{(\rho)\underset{\lambda}{|}n}$ into the previous equation according to (2a,b), we get

$$\begin{aligned} N^{\alpha}_{(\sigma)\underset{\theta}{|}m\underset{\lambda}{|}n} = & - e_{(\sigma)}\, g^{\underline{ps}} \underset{\theta}{\Omega}_{(\sigma)sm\underset{\lambda}{|}n} t^{\alpha}_{p} - \\ & - e_{(\sigma)}\, g^{\underline{ps}} \underset{\theta}{\Omega}_{(\sigma)sm} \big(\underset{\lambda}{\Phi}^{q}_{pn} t^{\alpha}_{q} + \sum_{\rho} \underset{\lambda}{\Omega}_{(\rho)pn} N^{\alpha}_{(\rho)} \big) + \\ & + \sum_{\rho} \underset{\theta}{\Psi}_{(\rho\sigma)m\underset{\lambda}{|}n} N^{\alpha}_{(\rho)} + \sum_{\rho} \underset{\theta}{\Psi}_{(\rho\sigma)m} \big(- e_{(\rho)}\, g^{\underline{ps}} \underset{\lambda}{\Omega}_{(\rho)sn} t^{\alpha}_{p} + \sum_{\tau} \underset{\lambda}{\Psi}_{(\tau\rho)n} N^{\alpha}_{(\tau)} \big), \end{aligned}$$

i.e.

$$
(27)\qquad
\begin{aligned}
N^{\alpha}_{(\sigma)\underset{\theta}{|}m\underset{\lambda}{|}n}=&\Big[-e_{(\sigma)}\,g^{\underline{ps}}\,\underset{\theta}{\Omega}_{(\sigma)sm\underset{\lambda}{|}n}-e_{(\sigma)}\,g^{\underline{qs}}\,\underset{\theta}{\Omega}_{(\sigma)qm}\,\underset{\lambda}{\Phi}{}^{p}_{sn}-\\
&-\sum_{\rho}e_{(\rho)}\,g^{\underline{ps}}\,\underset{\theta}{\Psi}_{(\rho\sigma)m}\,\underset{\lambda}{\Psi}_{(\rho)sn}\Big]\,t^{\alpha}_{p}+\\
&+\sum_{\rho}\Big[-e_{(\sigma)}\,g^{\underline{ps}}\,\underset{\theta}{\Omega}_{(\sigma)pm}\,\underset{\lambda}{\Omega}_{(\rho)sn}+\underset{\theta}{\Psi}_{(\rho\sigma)m\underset{\lambda}{|}n}+\sum_{\tau}\underset{\theta}{\Psi}_{(\tau\sigma)m}\,\underset{\lambda}{\Psi}_{(\rho\tau)n}\Big]N^{\alpha}_{(\rho)}\,.
\end{aligned}
$$

From here one gets

$$
(28)\qquad
\begin{aligned}
N^{\alpha}_{(\sigma)\underset{\theta}{|}m\underset{\lambda}{|}n}-N^{\alpha}_{(\sigma)\underset{\lambda}{|}n\underset{\theta}{|}m}=&\Big[-e_{(\sigma)}\,g^{\underline{ps}}\big(\underset{\theta}{\Omega}_{(\sigma)sm\underset{\lambda}{|}n}-\underset{\lambda}{\Omega}_{(\sigma)sn\underset{\theta}{|}m}\big)+\\
&+e_{(\sigma)}\,g^{\underline{qs}}\big(\underset{\theta}{\Phi}{}^{p}_{sm}\,\underset{\lambda}{\Omega}_{(\sigma)qn}-\underset{\lambda}{\Phi}{}^{p}_{sn}\,\underset{\theta}{\Omega}_{(\sigma)qm}\big)+\\
&+g^{\underline{ps}}\sum_{\rho}e_{(\rho)}\big(\underset{\theta}{\Omega}_{(\rho)sm}\,\underset{\lambda}{\Psi}_{(\rho\sigma)n}-\underset{\lambda}{\Omega}_{(\rho)sn}\,\underset{\theta}{\Psi}_{(\rho\sigma)m}\big)\Big]\,t^{\alpha}_{p}+\\
&+\sum_{\rho}\Big[\underset{\theta}{\Psi}_{(\rho\sigma)m\underset{\lambda}{|}n}-\underset{\lambda}{\Psi}_{(\rho\sigma)n\underset{\theta}{|}m}+e_{(\sigma)}\,g^{\underline{ps}}\big(\underset{\theta}{\Omega}_{(\rho)pm}\,\underset{\lambda}{\Omega}_{(\sigma)sn}-\\
&-\underset{\lambda}{\Omega}_{(\rho)pn}\,\underset{\theta}{\Omega}_{(\sigma)sm}\big)+\sum_{\tau}\big(\underset{\theta}{\Psi}_{(\tau\sigma)m}\,\underset{\lambda}{\Psi}_{(\rho\tau)n}-\underset{\lambda}{\Psi}_{(\tau\sigma)n}\,\underset{\theta}{\Psi}_{(\rho\tau)m}\big)\Big]N^{\alpha}_{(\rho)}\,.
\end{aligned}
$$

2.1. For $\theta=\lambda=1$ from the previous equation and (26a), (2b), we get *the 1st integrability condition of the 2nd derivational formula (2b)*

$$
(29)\qquad
\begin{aligned}
&\underset{1}{R}{}^{\alpha}_{\pi\mu\nu}\,N^{\pi}_{(\sigma)}\,t^{\mu}_{m}\,t^{\nu}_{n}-2\,\underset{\vee}{\Gamma}{}^{q}_{mn}\big(-e_{(\sigma)}\,g^{\underline{ps}}\,\underset{1}{\Omega}_{(\sigma)sq}\,t^{\alpha}_{p}+\sum_{\rho}\underset{1}{\Psi}_{(\rho\sigma)q}\,N^{\alpha}_{(\rho)}\big)=\\
&=\Big[-e_{(\sigma)}\,g^{\underline{ps}}\big(\underset{1}{\Omega}_{(\sigma)sm\underset{1}{|}n}-\underset{1}{\Omega}_{(\sigma)sn\underset{1}{|}m}\big)+e_{(\sigma)}\,g^{\underline{qs}}\big(\underset{1}{\Phi}{}^{p}_{sm}\,\underset{1}{\Omega}_{(\sigma)qn}-\\
&-\underset{1}{\Phi}{}^{p}_{sn}\,\underset{1}{\Phi}_{(\sigma)qm}\big)+g^{\underline{ps}}\sum_{\rho}e_{(\rho)}\big(\underset{1}{\Omega}_{(\rho)sm}\,\underset{1}{\Psi}_{(\rho\sigma)n}-\underset{1}{\Omega}_{(\rho)sn}\,\underset{1}{\Psi}_{(\rho\sigma)m}\big)\Big]\,t^{\alpha}_{p}+\\
&+\sum_{\rho}\Big[\underset{1}{\Psi}_{(\rho\sigma)m\underset{1}{|}n}-\underset{1}{\Psi}_{(\rho\sigma)n\underset{1}{|}m}+e_{(\sigma)}\,g^{\underline{ps}}\big(\underset{1}{\Omega}_{(\rho)pm}\,\underset{1}{\Omega}_{(\sigma)sn}-\\
&-\underset{1}{\Omega}_{(\rho)pn}\,\underset{1}{\Omega}_{(\sigma)sm}\big)+\sum_{\tau}\big(\underset{1}{\Psi}_{(\tau\sigma)m}\,\underset{1}{\Psi}_{(\rho\tau)n}-\underset{1}{\Psi}_{(\tau\sigma)n}\,\underset{1}{\Psi}_{(\rho\tau)m}\big)\Big]\,N^{\alpha}_{(\rho)}.
\end{aligned}
$$

Multiplying this equation by $a_{\underline{\alpha\beta}}\,t^{\beta}_{h}$, we obtain

$$
(30)\qquad
\begin{aligned}
&\underset{1}{R}_{\beta\pi\mu\nu}\,t^{\beta}_{h}\,N^{\pi}_{(\sigma)}\,t^{\mu}_{m}\,t^{\nu}_{n}+2\,e_{(\sigma)}\,\underset{\vee}{\Gamma}{}^{q}_{mn}\,\underset{1}{\Omega}_{(\sigma)hq}=\\
&=-e_{(\sigma)}\big(\underset{1}{\Omega}_{(\sigma)hm\underset{1}{|}n}-\underset{1}{\Omega}_{(\sigma)hn\underset{1}{|}m}\big)+e_{(\sigma)}\,g^{\underline{qs}}\big(\underset{1}{\Phi}_{hsm}\,\underset{1}{\Omega}_{(\sigma)qn}-\\
&-\underset{1}{\Phi}_{hsn}\,\underset{1}{\Omega}_{(\sigma)qm}\big)+\sum_{\rho}e_{(\rho)}\big(\underset{1}{\Omega}_{(\rho)hm}\,\underset{1}{\Psi}_{(\rho\sigma)n}-\underset{1}{\Omega}_{(\rho)hn}\,\underset{1}{\Psi}_{(\rho\sigma)m}\big).
\end{aligned}
$$

If in this equation we exchange the places of dummy indices β and π, take into account that (based on (2.9a) in [6]) $\underset{1}{R}_{\pi\beta\mu\nu} = -\underset{1}{R}_{\beta\pi\mu\nu}$, change h into i, σ into τ, ρ into σ and consider that, based on equation (39) in [5], $e_{(\sigma)} \underset{1}{\Psi}_{(\sigma\tau)m} = - e_{(\tau)} \underset{1}{\Psi}_{(\tau\sigma)m}$, and that from (27) in [5] the tensor $\underset{1}{\Phi}_{ijm}$ is antisymmetric on all pairs of indices, we get the equation (10). This means that the equation is still one form of the 1st Codazzi equation of the 1st kind.

If we multiply the equation (29) by $a_{\underline{\alpha\beta}} N^\beta_{(\varphi)}$, we get

$$
\begin{aligned}
&\underset{1}{R}_{\beta\pi\mu\nu} N^\beta_{(\varphi)} N^\pi_{(\sigma)} t^\mu_m t^\nu_n - 2 e_{(\varphi)} \Gamma^q_{\underset{\vee}{mn}} \Psi_{(\varphi\sigma)q} = \\
(31)\qquad &= e_{(\varphi)} \Big[\underset{1}{\Psi}_{(\varphi\sigma)m\underset{1}{|}n} - \underset{1}{\Psi}_{(\varphi\sigma)n\underset{1}{|}m} + e_{(\sigma)} g^{\underline{ps}} \big(\underset{1}{\Omega}_{(\varphi)pm} \underset{1}{\Omega}_{(\sigma)sn} - \underset{1}{\Omega}_{(\varphi)pn} \underset{1}{\Omega}_{(\sigma)sm} \big) + \\
&+ \sum_\tau \big(\underset{1}{\Psi}_{(\tau\sigma)m} \underset{1}{\Psi}_{(\varphi\tau)n} - \underset{1}{\Psi}_{(\tau\sigma)n} \underset{1}{\Psi}_{(\varphi\tau)m} \big) \Big],
\end{aligned}
$$

which represents the *2nd Coddazzi equation of the 1st kind.*

2.2. If we put in the equation (28) $\theta = \lambda = 2$ and take into account (26b) and (2b), we get *the 2nd integrability condition of the 2nd derivational formula*

$$
\begin{aligned}
&\underset{2}{R}^\alpha_{\pi\mu\nu} N^\pi_{(\sigma)} t^\mu_m t^\nu_n + 2 \Gamma^q_{\underset{\vee}{mn}} \big(- e_{(\sigma)} g^{\underline{ps}} \underset{2}{\Omega}_{(\sigma)sq} t^\alpha_p + \sum_\rho \underset{2}{\Psi}_{(\rho\sigma)q} N^\alpha_{(\rho)} \big) = \\
&= \Big[- e_{(\sigma)} g^{\underline{ps}} \big(\underset{2}{\Omega}_{(\sigma)sm\underset{2}{|}n} - \underset{2}{\Omega}_{(\sigma)sn\underset{2}{|}m} \big) + e_{(\sigma)} g^{\underline{qs}} \big(\underset{2}{\Phi}^p_{sm} \underset{2}{\Omega}_{(\sigma)qn} - \underset{2}{\Phi}^p_{sn} \underset{2}{\Omega}_{(\sigma)qm} \big) + \\
(32)\qquad &+ g^{\underline{ps}} \sum_\rho e_{(\rho)} \big(\underset{2}{\Omega}_{(\rho)sm} \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\rho)sn} \underset{2}{\Psi}_{(\rho\sigma)m} \big) \Big] t^\alpha_p + \\
&+ \sum_\rho \Big[\underset{2}{\Psi}_{(\rho\sigma)m\underset{2}{|}n} - \underset{2}{\Psi}_{(\rho\sigma)n\underset{2}{|}m} + e_{(\sigma)} g^{\underline{ps}} \big(\underset{2}{\Omega}_{(\rho)pm} \underset{2}{\Omega}_{(\sigma)sn} - \\
&- \underset{2}{\Omega}_{(\rho)pn} \underset{2}{\Omega}_{(\sigma)sm} \big) + \sum_\tau \big(\underset{2}{\Psi}_{(\tau\sigma)m} \underset{2}{\Psi}_{(\rho\tau)n} - \underset{2}{\Psi}_{(\tau\sigma)n} \underset{2}{\Psi}_{(\rho\tau)m} \big) \Big] N^\alpha_{(\rho)}.
\end{aligned}
$$

Multiplying this equation by $a_{\underline{\alpha\beta}} t^\beta_h$, we obtain

$$
\begin{aligned}
&\underset{2}{R}_{\beta\pi\mu\nu} t^\beta_h t^\mu_m t^\nu_n N^\pi_{(\sigma)} - 2 e_{(\sigma)} \Gamma^q_{\underset{\vee}{mn}} \underset{2}{\Omega}_{(\sigma)hq} = \\
(33)\qquad &= - e_{(\sigma)} \big(\underset{2}{\Omega}_{(\sigma)hm\underset{2}{|}n} - \underset{2}{\Omega}_{(\sigma)hn\underset{2}{|}m} \big) + e_{(\sigma)} g^{\underline{qs}} \big(\underset{2}{\Phi}_{hsm} \underset{2}{\Omega}_{(\sigma)qn} - \\
&- \underset{2}{\Phi}_{hsn} \underset{2}{\Omega}_{(\sigma)qm} \big) + \sum_\rho e_{(\rho)} \big(\underset{2}{\Omega}_{(\rho)hm} \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\rho)hn} \underset{2}{\Psi}_{(\rho\sigma)m} \big).
\end{aligned}
$$

As in the case of the equation (30), here one can prove that the equation (33) is equivalent to the equation (13), i.e. the equation (33) is still one form of the 1st Coddazzi equation of the 2*nd* kind.

Multiplying the equation (32) by $a_{\underline{\alpha\beta}} N^{\beta}_{(\rho)}$, we get *the 2nd Codazzi equation of the 2nd kind*

(34)
$$\begin{aligned} &\underset{2}{R}_{\beta\pi\mu\nu} N^{\beta}_{(\varphi)} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} + 2 e_{(\varphi)} \underset{\vee}{\Gamma}^{q}_{mn} \underset{2}{\Psi}_{(\varphi\sigma)q} = \\ &= e_{(\varphi)} \Big[\underset{2}{\Psi}_{(\varphi\sigma)m\underset{2}{|}n} - \underset{2}{\Psi}_{(\varphi\sigma)n\underset{2}{|}m} + e_{(\sigma)} g^{\underline{ps}} \big(\underset{2}{\Omega}_{(\varphi)pm} \underset{2}{\Omega}_{(\sigma)sn} - \\ &- \underset{2}{\Omega}_{(\varphi)pn} \underset{2}{\Omega}_{(\sigma)sm} \big) + \sum_{\tau} \big(\underset{2}{\Psi}_{(\tau\sigma)m} \underset{2}{\Psi}_{(\varphi\tau)n} - \underset{2}{\Psi}_{(\tau\sigma)n} \underset{2}{\Psi}_{(\varphi\tau)m} \big) \Big] . \end{aligned}$$

2.3. Putting in (28) $\theta = 1$. $\lambda = 2$, and using (26c), we obtain the *3rd integrability condition of the 2nd derivational formula*

(35)
$$\begin{aligned} &\underset{3}{R}^{\alpha}_{\pi mn} N^{\pi}_{(\sigma)} = \Big[- e_{(\sigma)} g^{\underline{ps}} \big(\underset{1}{\Omega}_{(\sigma)sm\underset{2}{|}n} - \underset{2}{\Omega}_{(\sigma)sn\underset{1}{|}m} \big) + \\ &+ e_{(\sigma)} g^{\underline{qs}} \big(\underset{1}{\Phi}^{p}_{sm} \underset{2}{\Omega}_{(\sigma)qn} - \underset{2}{\Phi}^{p}_{sn} \underset{2}{\Omega}_{(\sigma)qm} \big) + \\ &+ g^{\underline{ps}} \sum_{\rho} e_{(\rho)} \big(\underset{1}{\Omega}_{(\rho)sm} \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\rho)sn} \underset{1}{\Psi}_{(\rho\sigma)m} \big) \Big] t^{\alpha}_{p} + \\ &+ \sum_{\rho} \underset{1}{\Psi}_{(\rho\sigma)m\underset{2}{|}n} - \underset{2}{\Psi}_{(\rho\sigma)n\underset{1}{|}m} + e_{(\sigma)} g^{\underline{ps}} \big(\underset{1}{\Omega}_{(\rho)pm} \underset{2}{\Omega}_{(\sigma)sn} - \\ &- \underset{2}{\Omega}_{(\rho)pn} \underset{1}{\Omega}_{(\sigma)sm} \big) + \sum_{\tau} \big(\underset{1}{\Psi}_{(\tau\sigma)m} \underset{2}{\Psi}_{(\rho\tau)n} - \underset{2}{\Psi}_{(\tau\sigma)n} \underset{1}{\Psi}_{(\rho\tau)m} \big) \Big] N^{\alpha}_{(\rho)} . \end{aligned}$$

Multiplying this equation by $a_{\underline{\alpha\beta}} t^{\beta}_{h}$, we get

(36)
$$\begin{aligned} &\underset{3}{R}_{\beta\pi mn} t^{\beta}_{h} N^{\pi}_{(\sigma)} = - e_{(\sigma)} \big(\underset{1}{\Omega}_{(\sigma)hm\underset{2}{|}n} - \underset{2}{\Omega}_{(\sigma)hn\underset{1}{|}m} \big) + \\ &+ e_{(\sigma)} g^{\underline{qs}} \big(\underset{1}{\Phi}_{hsm} \underset{2}{\Omega}_{(\sigma)qn} - \underset{2}{\Phi}_{hsn} \underset{1}{\Omega}_{(\sigma)qm} \big) + \\ &+ \sum_{\rho} e_{(\rho)} \big(\underset{1}{\Omega}_{(\rho)hm} \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\rho)hn} \underset{1}{\Psi}_{(\rho\sigma)m} \big) . \end{aligned}$$

Taking into account that on the the basis of (4.4a) in [6] $\underset{3}{R}_{\pi\beta mn} = - \underset{3}{R}_{\beta\pi mn}$, we prove by the similar procedure as in the case of the equation (30), that the equation (36) is still one form of the 1st Codazzi equation of the 3rd kind.

Multiplying the equation (35) by $a_{\underline{\alpha\beta}} N^{\beta}_{(\varphi)}$, we get *the 2nd Codazzi equation of the 3rd kind*

(37)
$$\begin{aligned} &\underset{3}{R}_{\beta\pi mn} N^{\beta}_{(\varphi)} N^{\pi}_{(\sigma)} = e_{(\varphi)} \Big[\underset{1}{\Psi}_{(\varphi\sigma)m\underset{2}{|}n} - \underset{2}{\Psi}_{(\varphi\sigma)n\underset{1}{|}m} + \\ &+ e_{(\sigma)} g^{\underline{ps}} \big(\underset{1}{\Omega}_{(\varphi)pm} \underset{2}{\Omega}_{(\sigma)sn} - \underset{2}{\Omega}_{(\varphi)pn} \underset{1}{\Omega}_{(\sigma)sm} \big) + \\ &+ \sum_{\tau} \big(\underset{1}{\Psi}_{(\tau\sigma)m} \underset{2}{\Psi}_{(\varphi\tau)n} - \underset{2}{\Psi}_{(\tau\sigma)n} \underset{1}{\Psi}_{(\varphi\tau)m} \big) \Big] . \end{aligned}$$

2.4. If in (28) we take $\theta=\lambda=3$ and use (26d) and (2b), we shall get *the 4 th integrability condition of the 2 nd derivational* formula

$$(38)\qquad \begin{aligned} &\underset{1}{R}{}^{\alpha}_{\pi\mu\nu} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} + 2\,\Gamma^{q}_{\underset{\vee}{mn}} \Big(-e_{(\sigma)} g^{\underline{ps}} \underset{3}{\Omega}_{(\sigma)sq}\, t^{\alpha}_{q} + \sum_{\rho} \underset{3}{\Psi}_{(\rho\sigma)q} N^{\alpha}_{(\rho)} \Big) = \\ &= \Big[-e_{(\sigma)} g^{\underline{ps}} \big(\underset{3}{\Omega}_{(\sigma)sm\underset{3}{|}n} - \underset{3}{\Omega}_{(\sigma)sn\underset{3}{|}m} \big) + e_{(\sigma)}\, g^{\underline{qs}} \big(\underset{3}{\Phi}{}^{p}_{sm} \underset{3}{\Omega}_{(\sigma)qn} - \\ &- \underset{3}{\Phi}{}^{p}_{sn} \underset{3}{\Omega}_{(\sigma)qm} \big) + g^{\underline{ps}} \sum_{\rho} e_{(\rho)} \big(\underset{3}{\Omega}_{(\rho)sm} \underset{3}{\Psi}_{(\rho\sigma)n} - \underset{3}{\Omega}_{(\rho)sn} \underset{3}{\Psi}_{(\rho\sigma)m} \big) \Big] t^{\alpha}_{p} + \\ &+ \sum_{\rho} \Big[\underset{3}{\Psi}_{(\rho\sigma)m\underset{3}{|}n} - \underset{3}{\Psi}_{(\rho\sigma)n\underset{3}{|}m} + e_{(\sigma)}\, g^{\underline{ps}} \big(\underset{3}{\Omega}_{(\rho)pm} \underset{3}{\Omega}_{(\sigma)sn} - \\ &- \underset{3}{\Omega}_{(\rho)pn} \underset{3}{\Omega}_{(\sigma)sm} \big) + \sum_{\tau} \big(\underset{3}{\Psi}_{(\tau\sigma)m} \underset{3}{\Psi}_{(\rho\tau)n} - \underset{3}{\Psi}_{(\tau\sigma)n} \underset{3}{\Psi}_{(\rho\tau)m} \big) \Big] N^{\alpha}_{(\rho)}\,. \end{aligned}$$

Using (3′) and (3″) we can write the previous equation in another form. Multiplying the obtained equation by $a^{\underline{\alpha\beta}} t^{\beta}_{h}$, we get

$$(39)\qquad \begin{aligned} &\underset{1}{R}_{\beta\pi\mu\nu}\, t^{\beta}_{h} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} - 2\, e_{(\sigma)} \Gamma^{q}_{\underset{\vee}{mn}} \Omega_{(\sigma)hq} = \\ &= -e_{(\sigma)} \big(\underset{1}{\Omega}_{(\sigma)hm\underset{2}{|}n} - \underset{1}{\Omega}_{(\sigma)hn\underset{2}{|}m} \big) + e_{(\sigma)}\, g^{\underline{qs}} \big(\underset{3}{\Phi}_{hsm} \underset{1}{\Omega}_{(\sigma)qn} - \\ &- \underset{3}{\Phi}_{hsn} \underset{1}{\Omega}_{(\sigma)qm} \big) + \sum_{\rho} e_{(\rho)} \big(\underset{1}{\Omega}_{(\rho)hm} \underset{1}{\Psi}_{(\rho\sigma)n} - \underset{1}{\Omega}_{(\rho)hn} \underset{1}{\Psi}_{(\rho\sigma)m} \big)\,. \end{aligned}$$

As in previous cases, one can prove that the equation (39) is equivalent to the equation (19) and to the equation (10), i.e., the equation (39) is still one form of the 1st Codazzi equation of the 4 th kind (1st kind). All these equations reduce to the equation (19′).

Multiplying (38) by $a_{\alpha\beta} N^{\beta}_{(\varphi)}$, we obtain *the 2nd Codazzi equation of the 4 th kind*

$$(40)\qquad \begin{aligned} &\underset{1}{R}_{\beta\pi\mu\nu} N^{\beta}_{(\varphi)} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} + 2\, e_{(\varphi)} \Gamma^{q}_{\underset{\vee}{mn}} \underset{1}{\Psi}_{(\varphi\sigma)q} = \\ &= e_{(\varphi)} \Big[\underset{1}{\Psi}_{(\varphi\sigma)m\underset{2}{|}n} - \underset{1}{\Psi}_{(\varphi\sigma)n\underset{2}{|}m} + \\ &+ e_{(\sigma)}\, g^{\underline{ps}} \big(\underset{1}{\Omega}_{(\varphi)pm} \underset{1}{\Omega}_{(\sigma)sn} - \underset{1}{\Omega}_{(\varphi)pn} \underset{1}{\Omega}_{(\sigma)sm} \big) + \\ &+ \sum_{\tau} \big(\underset{1}{\Psi}_{(\tau\sigma)m} \underset{1}{\Psi}_{(\varphi\tau)n} - \underset{1}{\Psi}_{(\tau\sigma)n} \underset{1}{\Psi}_{(\varphi\tau)m} \big) \Big]. \end{aligned}$$

Using the results

$$(41\text{a})\qquad \underset{1}{\Psi}_{(\varphi\sigma)m\underset{1}{|}n} - \underset{1}{\Psi}_{(\varphi\sigma)n\underset{1}{|}m} = \underset{1}{\Psi}_{(\varphi\sigma)m,n} - \underset{1}{\Psi}_{(\varphi\sigma)n,m} - 2\,\Gamma^{r}_{\underset{\vee}{mn}} \underset{1}{\Psi}_{(\varphi\sigma)r}\,,$$

$$(41\text{b})\qquad \underset{1}{\Psi}_{(\varphi\sigma)m\underset{2}{|}n} - \underset{1}{\Psi}_{(\varphi\sigma)n\underset{2}{|}m} = \underset{1}{\Psi}_{(\varphi\sigma)m,n} - \underset{1}{\Psi}_{(\varphi\sigma)n,m} + 2\,\Gamma^{r}_{\underset{\vee}{mn}} \underset{1}{\Psi}_{(\varphi\sigma)r}\,,$$

we reduce the equation (39) and (40) to the form

$$
\begin{aligned}
&\underset{1}{R}_{\beta\pi\mu\nu} N^{\beta}_{(\varphi)} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} = \\
(40')\quad &= e_{(\varphi)} \big[\underset{1}{\Psi}_{(\varphi\sigma)m,n} - \underset{1}{\Psi}_{(\varphi\sigma)n,m} + e_{(\sigma)} \underline{g}^{ps} \big(\underset{1}{\Omega}_{(\varphi)pm} \underset{1}{\Omega}_{(\sigma)sn} - \\
&- \underset{1}{\Omega}_{(\varphi)pn} \underset{1}{\Omega}_{(\sigma)sm} \big) + \sum_{\tau} \big(\underset{1}{\Psi}_{(\tau\sigma)m} \underset{1}{\Psi}_{(\varphi\tau)n} - \underset{1}{\Psi}_{(\tau\sigma)n} \underset{1}{\Psi}_{(\varphi\tau)m} \big) \big],
\end{aligned}
$$

i.e. the equations (31), (40) and (40′) are equivalent and represent various forms of the 2*nd* Codazzi equation of ihe 1st kind.

2.5. Taking in (28) $\theta=\lambda=4$ considering (26e) and (2b) we get *the 5 th integrability condition of the 2nd derivational formula*

$$
\begin{aligned}
&\underset{2}{R}{}^{\alpha}_{\pi\mu\nu} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} - 2 \underset{\vee}{\Gamma}{}^{q}_{mn} \big(- e_{(\sigma)} \underline{g}^{ps} \underset{4}{\Omega}_{(\sigma)sq} t^{\alpha}_{p} + \sum_{\rho} \underset{4}{\Psi}_{(\rho\sigma)q} N^{\alpha}_{(\rho)} \big) = \\
&= \big[- e_{(\sigma)} \underline{g}^{ps} \big(\underset{4}{\Omega}_{(\sigma)sm\,|\,\underset{4}{n}} - \underset{4}{\Omega}_{(\sigma)sn\,|\,\underset{4}{m}} \big) + e_{(\sigma)} \underline{g}^{qs} \big(\underset{4}{\Phi}{}^{p}_{sm} \underset{4}{\Omega}_{(\sigma)qn} - \underset{4}{\Phi}{}^{p}_{sn} \underset{4}{\Omega}_{(\sigma)qm} \big) + \\
(42)\quad &+ \underline{g}^{ps} \sum_{\rho} e_{(\rho)} \big(\underset{4}{\Omega}_{(\rho)sm} \underset{4}{\Psi}_{(\rho\sigma)n} - \underset{4}{\Omega}_{(\rho)sn} \underset{4}{\Psi}_{(\rho\sigma)m} \big) \big] t^{\alpha}_{p} + \\
&+ \sum_{\rho} \big[\underset{4}{\Psi}_{(\rho\sigma)m\,|\,\underset{4}{n}} - \underset{4}{\Psi}_{(\rho\sigma)n\,|\,\underset{4}{m}} + e_{(\sigma)} \underline{g}^{ps} \big(\underset{4}{\Omega}_{(\rho)pm} \underset{4}{\Omega}_{(\sigma)sn} - \underset{4}{\Omega}_{(\rho)pn} \underset{4}{\Omega}_{(\sigma)sm} \big) + \\
&+ \sum_{\tau} \big(\underset{4}{\Psi}_{(\tau\sigma)m} \underset{4}{\Psi}_{(\rho\tau)n} - \underset{4}{\Psi}_{(\tau\sigma)n} \underset{4}{\Psi}_{(\rho\tau)m} \big) \big] N^{\alpha}_{(\rho)} .
\end{aligned}
$$

With regard to (3′, 3″), multiplying the previous equation by $a_{\underline{\alpha\beta}}\, t^{\beta}_{h}$, we get

$$
\begin{aligned}
&\underset{2}{R}_{\beta\pi\mu\nu}\, t^{\beta}_{h} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} + 2 \underset{\vee}{\Gamma}{}^{q}_{mn} \underset{2}{\Omega}_{(\sigma)hq} = \\
(43)\quad &= - e_{(\sigma)} \big(\underset{2}{\Omega}_{(\sigma)hm\,|\,\underset{1}{n}} - \underset{2}{\Omega}_{(\sigma)hn\,|\,\underset{1}{m}} \big) + e_{(\sigma)} \underline{g}^{qs} \big(\underset{4}{\Phi}_{hsm} \underset{2}{\Omega}_{(\sigma)qn} - \\
&- \underset{4}{\Phi}_{hsn} \underset{2}{\Omega}_{(\sigma)qm} \big) + \sum_{\rho} e_{(\rho)} \big(\underset{2}{\Omega}_{(\rho)hm} \underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\rho)hn} \underset{2}{\Psi}_{(\rho\sigma)m} \big) .
\end{aligned}
$$

This equation is equivalent to the equation (22) and (13), i.e., we obtained one more form of the 1st Codazzi equation of the 2*nd* kind.

Multiplying the equation (42) by $a_{\underline{\alpha\beta}}\, N^{\beta}_{(\varphi)}$, we get *the 2 nd Codazzi equation of the 5 th kind*

$$
\begin{aligned}
&\underset{2}{R}_{\beta\pi\mu\nu} N^{\beta}_{(\varphi)} N^{\pi}_{(\sigma)} t^{\mu}_{m} t^{\nu}_{n} - 2 e_{(\varphi)} \underset{\vee}{\Gamma}{}^{q}_{mn} \underset{2}{\Psi}_{(\varphi\sigma)q} = \\
&= e_{(\varphi)} \big[\underset{2}{\Psi}_{(\varphi\sigma)m\,|\,\underset{1}{n}} - \underset{2}{\Psi}_{(\varphi\sigma)n\,|\,\underset{1}{m}} + \\
(44)\quad &+ e_{(\sigma)} \underline{g}^{ps} \big(\underset{2}{\Omega}_{(\varphi)pm} \underset{2}{\Omega}_{(\sigma)sn} - \underset{2}{\Omega}_{(\varphi)pn} \underset{2}{\Omega}_{(\sigma)sm} \big) + \\
&+ \sum_{\tau} \big(\underset{2}{\Psi}_{(\tau\sigma)m} \underset{2}{\Psi}_{(\varphi\tau)n} - \underset{2}{\Psi}_{(\tau\sigma)n} \underset{2}{\Psi}_{(\varphi\tau)m} \big) \big].
\end{aligned}
$$

Taking into account that

$$\underset{2}{\Psi}_{(\varphi\sigma)m\underset{2}{|}n} - \underset{2}{\Psi}_{(\varphi\sigma)n\underset{2}{|}m} = \underset{2}{\Psi}_{(\varphi\sigma)m,n} - \underset{2}{\Psi}_{(\varphi\sigma)n,m} + 2\,\Gamma^{r}_{\underset{\vee}{mn}}\,\underset{2}{\Psi}_{(\varphi\sigma)r}, \tag{45a}$$

$$\underset{2}{\Psi}_{(\varphi\sigma)m\underset{1}{|}n} - \underset{2}{\Psi}_{(\varphi\sigma)n\underset{1}{|}m} = \underset{2}{\Psi}_{(\varphi\sigma)m,n} - \underset{2}{\Psi}_{(\varphi\sigma)n,m} - 2\,\Gamma^{r}_{\underset{\vee}{mn}}\,\underset{2}{\Psi}_{(\varphi\sigma)r}, \tag{45b}$$

the equations (34) and (44) can be writen in the form

$$\begin{aligned} &\underset{2}{R}_{\beta\pi\mu\nu}\, N^{\beta}_{(\varphi)}\, N^{\pi}_{(\sigma)}\, t^{\mu}_{m}\, t^{\nu}_{n} = \\ &= e_{(\varphi)}\Big[\underset{2}{\Psi}_{(\varphi\sigma)m,n} - \underset{2}{\Psi}_{(\varphi\sigma)n,m} + e_{(\sigma)}\, \underline{g}^{ps}\big(\underset{2}{\Omega}_{(\varphi)pm}\underset{2}{\Omega}_{(\sigma)sn} - \\ &- \underset{2}{\Omega}_{(\varphi)pn}\underset{2}{\Omega}_{(\sigma)sm}\big) + \sum_{\tau}\big(\underset{2}{\Psi}_{(\tau\sigma)m}\underset{2}{\Psi}_{(\varphi\tau)n} - \underset{2}{\Psi}_{(\tau\sigma)n}\underset{2}{\Psi}_{(\varphi\tau)m}\big)\Big], \end{aligned} \tag{44'}$$

i.e., the equations (34), (44), and (44′) are equivalent and represent various forms of the $2\,nd$ Codazzi equation of the $2nd$ kind.

2.6. Putting in (28) $\theta=3$, $\lambda=4$ and using (26 f), we obtain *the 6 th integrability condition of the 2 nd derivational formula*

$$\begin{aligned} &\underset{4}{R}^{\alpha}_{\pi mn} N^{\pi}_{(\sigma)} = \Big[-e_{(\sigma)}\, \underline{g}^{ps}\big(\underset{3}{\Omega}_{(\sigma)sm\underset{4}{|}n} - \underset{4}{\Omega}_{(\sigma)sn\underset{3}{|}m}\big) + \\ &+ e_{(\sigma)}\, \underline{g}^{qs}\big(\underset{3}{\Phi}^{p}_{sm}\underset{4}{\Omega}_{(\sigma)qn} - \underset{4}{\Phi}^{p}_{sn}\underset{3}{\Omega}_{(\sigma)qm}\big) + \\ &+ \underline{g}^{ps}\sum_{\rho} e_{(\rho)}\big(\underset{3}{\Omega}_{(\rho)sm}\,\underset{4}{\Psi}_{(\rho\sigma)n} - \underset{4}{\Omega}_{(\rho)sn}\,\underset{3}{\Psi}_{(\rho\sigma)m}\big)\Big]\, t^{\alpha}_{p} + \\ &+ \sum_{\rho}\Big[\underset{3}{\Psi}_{(\rho\sigma)m\underset{4}{|}n} - \underset{4}{\Psi}_{(\rho\sigma)n\underset{3}{|}m} + e_{(\sigma)}\, \underline{g}^{ps}\big(\underset{3}{\Omega}_{(\rho)pm}\underset{4}{\Omega}_{(\sigma)sn} - \\ &- \underset{4}{\Omega}_{(\rho)pn}\,\underset{3}{\Omega}_{(\sigma)sm}\big) + \sum_{\tau}\big(\underset{3}{\Psi}_{(\tau\sigma)m}\,\underset{4}{\Psi}_{(\rho\tau)n} - \underset{4}{\Psi}_{(\tau\sigma)n}\,\underset{3}{\Psi}_{(\rho\tau)m}\big)\Big] N^{\alpha}_{(\rho)}. \end{aligned} \tag{46}$$

Using (3′, 3″) and multiplying by $\underline{a}_{\alpha\beta}\, t^{\beta}_{h}$ we get

$$\begin{aligned} &\underset{4}{R}_{\beta\pi mn}\, t^{\beta}_{h}\, N^{\pi}_{(\sigma)} = -e_{(\sigma)}\big(\underset{1}{\Omega}_{(\sigma)hm\underset{1}{|}n} - \underset{2}{\Omega}_{(\sigma)hn\underset{2}{|}m}\big) + \\ &+ e_{(\sigma)}\, \underline{g}^{qs}\big(\underset{3}{\Phi}_{hsm}\,\underset{2}{\Omega}_{(\sigma)qn} - \underset{4}{\Phi}_{hsn}\,\underset{1}{\Omega}_{(\sigma)qm}\big) + \\ &+ \sum_{\rho} e_{(\rho)}\big(\underset{1}{\Omega}_{(\rho)hm}\,\underset{2}{\Psi}_{(\rho\sigma)n} - \underset{2}{\Omega}_{(\rho)hn}\,\underset{1}{\Psi}_{(\rho\sigma)m}\big). \end{aligned} \tag{47}$$

Since from (4.4a) in [6] it follows that $\underset{4}{R}_{\pi\beta mn} = -\underset{4}{R}_{\beta\pi mn}$, the equation (47) is equivalent to the equation (25) i.e., (25) and (47) are two forms of the 1st Coddazzi equation of the $6\,th$ kind.

If we multiply the equation (46) by $a_{\underline{\alpha\beta}} N^{\beta}_{(\varphi)}$, we get *the 2nd Codazzi equation of the 6th kind*

$$R_{\underset{4}{\beta\pi mn}} N^{\beta}_{(\varphi)} N^{\pi}_{(\sigma)} =$$

(48)
$$= e_{(\varphi)} \big[\underset{1}{\Psi}_{(\varphi\sigma)m\underset{1}{|}n} - \underset{2}{\Psi}_{(\varphi\sigma)n\underset{2}{|}m} + e_{(\sigma)}\, g^{\underline{ps}} \big(\underset{1}{\Omega}_{(\varphi)pm} \underset{2}{\Omega}_{(\sigma)sn} -$$

$$- \underset{2}{\Omega}_{(\varphi)pn} \underset{1}{\Omega}_{(\sigma)sm} \big) + \sum_{\tau} \big(\underset{1}{\Psi}_{(\tau\sigma)m} \underset{2}{\Psi}_{(\varphi\tau)n} - \underset{2}{\Psi}_{(\tau\sigma)n} \underset{1}{\Psi}_{(\varphi\tau)m} \big) \big].$$

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