PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 31 (45) 1982, 109—120

AROUND THE NUMBER OF CHAINS IN PARTITIVE SETS*)**)

Đuro Kurepa

0. Generalities

0:0. Among crucial and the most fruitful formations of sets is the one of the partitive set PS for any given set S; also for a given cardinal (ordinal) number n one designs by Pn or $P(n)$ any partitive set PS such that $n=kS$ (= cardinal number of S). The set Pn is ordered by $\subset$ (the relation $\subset$ includes the case of = as well).

0:1. The question arises about the set $l\,Pn\,[LP(n)]$ of all subchains [maximal subchains] of $(P(n), \subset)$; of course, $v\in lP(n)$ (v denotes the empty or vacuous set). The cases of n finite and n transfinite behave quite distinctly, in particular as concerns LPn; as a matter of fact, one proves easily that $klPu=n!$ for any $n\in\{0, 1, 2, \ldots\}$; on the other side, one knows that the assumption $LPn\neq v$ for every cardinal n is independent of the usual axioms of the theory of sets and is equivalent to the following proposition.

OP (Ordering principle) Every set is totally orderable.[1]

0:2. One has $kP\aleph_\alpha=2^{\aleph_\alpha}$, $lP\aleph_\alpha\subset P^2\aleph_\alpha$ thus $klP\aleph_\alpha\leqslant \exp_2\aleph_\alpha$. Does here $\leqslant$ mean = ?

---

* Financially supported by Republička zajednica za nauku SR Srbije through Matematički institut Beograd.

** Partly presented in: 1) Zagreb 1954:03:17, O kombinacijama [On combinations] (v. Glasnik Mat. fiz. astr. 9:1 (Zagreb 1954) p. 73; 2) Mengentheoretische Kombinatorik (presented on 1968:02:12 at the Kongress der Mathematiker der DDR (Rostock 1968:02:12 —17); 3) Matematički institut, Beograd (the 1976:11:05:5); 4) On a sequence of integers (1976:06:29:2 at the 5th Hungarian Combinatorial Colloquium (Kesztely 1976:06:28—07:03); 5) Kolloquium in Erlangen 1976:11:09:2 Einige Resultate aus der Kombinatorik und der Graphentheorie.

[1] *OP* is strongly weaker than the proposition.

W (Well-ordering principle): Every set is well-orderable.
We proved that $W\Leftrightarrow OP\wedge MA$, where

MA Every ordered set contains a maximal antichain.

Let us observe that $W\Leftrightarrow MCE$ (Maximal Chain Extension Principle) $L(E, \leq)\neq v$ for every ordered set $(E, \leqslant)$ (F. Hausdorff). Of course, $MA\Leftrightarrow MAE$ (Maximal Antichain Extension). For every ordered set $(E, \leqslant)$ and every antichain $a\subset(E, \leqslant)$ there exists a maximal antichain A in $(E, \leqslant)$ such that $a\subset A$.

Also: $MCE\Leftrightarrow$ For every ordered set $(E, \leqslant)$ and every chain $l\subset(E, \leqslant)$ there is a maximal chain $L\subset(E, \leqslant)$ such that $l\subset L$.

0:2:1. Let us consider the following statements

$(kl)_m$ $$klPm = \exp_2 m\ (: = 2^{2^m})$$

(kl) For every infinite cardinal n the statement $(kl)_m$ holds.

$(K_1)_m$ $$K_1 m = 2^m$$

(K_1) For every infinite cardinal m one has $K_1 m = \exp m$, where

$K_1 m = \sup_l kl$, l running through systems of totally ordered sets such that $m = k_1 l := \inf_x kx$, each x being everywhere dense in l in the sense that each open non empty interval of l meets x.

As we proved elsewhere there is no restriction to assume that in the definition of $K_1 l$ one has $l \subset Pm$ (see. Kurepa 1957:1 Theorem 8:1).

0:2:2. Does $(K_1)_m \Rightarrow (kl)_m$ for a given infinite cardinal m? Yes, if $K_1 m$ is reached i.e. if there is a chain $l \subset Pm$ such that $kl = \exp m$ and $k_1 l = m$.

0:2:3. Let us observe that $GCH \Rightarrow (K_1)$. But the proposition (K_1) could be considered also independently of the GCH. E. g. every K-tree of cardinality $\aleph_1$ and with $\exp \aleph_1$ branches guarantees $(K_1)_{\aleph_1}$.

0:3. In the present paper we restrict ourselves to $P(n)$ for $n \in N$. In § 2 we reprove a result of M. Popadić concerning $klPn$ for any natural number and establish some connexions between summation formula on combinations and iterated differences. For numbers $l(n, r)$ for a given n, we establish that they are increasing at beginning and decreasing at the end. We are not able to prove the assumption that the sequence $l(n, r)$ $(r = 0, 1, \ldots)$ consists of a strictly increasing initial segment and of the strictly decreasing remainder and that there are no equal members in the r^{th} row (we guess that in the table of numbers $l(n, r)$ for $r \neq 0$ all numbers aıe pairwise distinct).

For numbers $l(n, r)\, n!^{-1}$ $(r = n, n-1, \ldots)$ we prove that they are n-polynomials of degrees 1, 2, 3, ... Is it so for every r?

In § 4 we formulate some problems on numbers $\Delta^r b^n$.

1. Differences of a function.

1:1. For a function $f(x)$ and $h \in R$ one puts $\Delta_h f(x) := f(x+h) - f(x)$, $\Delta_h^2 f(x) := \Delta_h f(x+h) - \Delta_h f(x), \ldots, \Delta_h^n f(x) = \Delta_h \Delta_h^{n-1} f(x)$. One puts also $\Delta_h^0 f(x) := f(x)$. One writes $\Delta_1^n f(x) := \Delta^n f(x)$.

1:2. Theorem. Let $f(x)$ be a real-valued function defined in a set $\{a, a+h, a+2h, \ldots\} \subset R$; then

$$\Delta_h^n f(a) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} f(a + (n-\nu)h). \tag{1:3}$$

In particular, for any power b^m one has

$$\Delta_h^n b^m = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} (b + (n-\nu)h)^m. \tag{1:4}$$

If $f(x)=g(x)\,k(x)$, then

$$\Delta_h^n\, g(x)\,k(x)=\sum_{i=0}^{n}\binom{n}{i}\Delta_h^i\, g(a)\,\Delta_h^{n-i}\,k(x+ih); \tag{1:5}$$

in particular (case: $h=1$, $n+m+1$, $g(x)=x$, $k(x)=x^{s-1}$) one has

$$\Delta^{m+1}a^s=(m+1+a)\,\Delta^{m+1}a^{s-1}+(m+1)\,\Delta^m a^{s-1}. \tag{1:6}$$

The proof is carried out by an induction argument.

1:7. Theorem.

$$\sum_{e=0}^{n}\binom{n}{e}\Delta^r a^e=\Delta^r(a+1)^n. \tag{1:8}$$

Proof. Since $\Delta^r a^e=\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r+a-i)^e$, one has

$$(1:8)_1 \qquad =\sum_{e=0}^{n}\binom{n}{e}\sum_{i=0}^{r}(-1)^r\binom{r}{i}(r+a-i)^e=$$

$$=\sum_{i=0}^{r}(-1)^i\binom{r}{i}\sum_{e=0}^{n}\binom{n}{e}(r+a-i)^e=$$

$$=\sum_{i=0}^{r}(-1)^i\binom{r}{i}(1+(r+a-i))^n=$$

$$=\sum_{i=0}^{r}(-1)^i\binom{r}{i}(a+1+r-i)^n=\Delta^r(a+1)^n=(1:8)_2.$$

1:9. Theorem. *Let a be any real or complex number; for every positive integer n and every $r\in I_n:=\{0,1,2,\dots,n-1\}$ one has*

$\Delta^n a^r=0$

$\Delta^n a^n=n!$ *In particular, $\Delta^{n+1}a^n=0$ for every natural number n.*

Proof. For $n=1$ one has $\Delta^1 1^1=2^1-1^1=1=1!$

$$\Delta^1 1^0=2^0-1^0=0.$$

Consequently, the statement is holding for $n=1$. Assume now that the statement holds for $n=m\geqslant 1$; we are going to prove that it is holding for $n=m+1$, $r=0,1,\dots,m$, as well. First of all, the statement is obviously holding for $n=m+1$, $r=0$. Assume that it holds also for $n=m+1$ and $r=0,1,2,\dots,s-1$, where $s-1<m$; we are going to prove the statement also for $n=m+1$, $r=s$. Now, we have the formula (1:6); by induction assumption, both summands (in 1:6) are 0; consequently, also $\Delta^{m+1}a^s=0$. In particular, for $m+1=s+1$ the equality (1:6) yields $\Delta^{m+1}a^{m+1}=(m+1+m+1)\,\Delta^{m+1}a^m+(m+1)\,\Delta^m a^m=$ $=$(the first summand is 0)$=(m+1)\,\Delta^m a^m=(m+1)\,m!=(m+1)!$.

1:10. Lemma. *For every 2-un (n, r) of natural numbers one has*

$$(1:11) \qquad \sum_{e=0}^{n} \binom{n}{e} \Delta^r 1^e = \Delta^r 2^n.$$

Proof. Since $\Delta^r 1^e = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (r+1-i)^e$, one has

$$(1:11)_1 \quad = \sum_{e=0}^{n} \binom{n}{e} \sum_{i=0}^{r} (-1)^i \binom{r}{i} (r+1-i)^e = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \sum_{e=0}^{n} \binom{n}{e} (r+1-i)^e =$$

$$= \sum_{i=0}^{r} (-1)^i \binom{r}{i} (1+(r+1-i))^n = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (r+2-i)^n = \Delta^r 2^n =$$

$$= (1:11)_2.$$

1:12. Lemma. *If (a, n) is any 2-un of natural numbers such that $a < n$, then for any integer $r \geqslant 0$*

$$(1:13) \qquad \sum_{a=r}^{n-1} \binom{n}{a} \Delta^r 1^a = \Delta^{r+1} 1^n; \text{ in particular for } r=0 \text{ one has}$$

$$(1:14) \qquad \sum_{a=0}^{n-1} \binom{n}{a} \Delta^0 1^a = \Delta 1^n, \text{ where } \Delta^0 1^a := 1^a.$$

Proof. At first, let us prove (1:13) for $r=0$, i.e. that (1:14) is holding. Now, $(1:14)_1 = \sum_{a=0}^{n} \binom{n}{a} 1^a - \binom{n}{n} \Delta^0 1^n = 2^n - 1^n := \Delta 1^n = (1:14)_2$.

If $r > 0$, then the summator in $(1:13)_1$ satisfies, symbolically,

$$\sum_{a=r}^{n-1} = \sum_{a=0}^{n} - \sum_{a=0}^{r-1} - \sum_{a=n};$$

therefore,

$(1:13)_1 \quad = \sum_{a=0}^{n} \binom{n}{a} \Delta^r 1^a - \sum_{a=0}^{r-1} \binom{n}{a} \Delta^r 1^a - \binom{n}{n} \Delta^r 1^n$; now, the first sum equals $\Delta^r 2^n$ (see 1:10 Lemma); the second sum equals 0 (v. 1:9 Lemma); therefore the last expression for $(1:13)_1$ yields

$$(1:13)_1 \quad = \Delta^r 2^n - \Delta^r 1^n = \Delta^r (2^n - 1^n) = \Delta^r \Delta 1^n = \Delta^{r+1} 1^n. \text{ Q.E.D.}$$

Remark. It is useful to compare the content of lemmas 1:10, 1:11; in either case, the left side is a scalar product; only the boundary of summations are distinct.

2. Set and number of paths of a graph.

2:1. For any graph $\mathcal{G}$ let $p\mathcal{G}$ or $l\mathcal{G}$ denote the set of all non empty paths or chains in $\mathcal{G}$.

The empty set is considered as a path (chain) in every graph.

2:2. For any cardinal number n let $p_n\mathcal{G}$ denote the set of all paths of $\mathcal{G}$ of cardinality n each. Consequently, $p\mathcal{G}=\bigcup_n p_n$ $(n=0, 1, 2, \ldots)$. In particular, $p_0\mathcal{G}$ denotes the set consisting of the empty path.

2:3. For any (n, r) of numbers we define $p(n, r):=kp_r P(n):=p_{nr}$. Consequently, p_{nr} is a cardinal number and not a system of sets. Of course, if $n<r$, then $p_{nr}=0$. How to determine the numbers $p(n, r)$?

2:4. Theorem. *Let* $0\leqslant r\leqslant n$ *and let* $e:=(e_0, e_1, \ldots, e_r)\in\binom{I_{1+n}}{1+r}$ *be any strictly increasing sequence of digits* $\in I_{1+n}:=\{0, 1, 2, \ldots, n\}$. *The number* $p(n, 1+r)$ *of* $(1+r)$ — *chains* $x=(x_0<x_1<\cdots<x_r)$ *satisfying* $k(x_i)=e_i$ *and* $x\subset P(n)$ *equals:* $\binom{n}{e_0}$ *for* $r=0$; *thus* $\binom{n}{0}=1$ *provided* $e_0=0$; *and*

$$n(e):=\binom{n}{e_r}\binom{e_r}{e_{r-1}}\binom{e_{r-1}}{e_{r-2}}\cdots\binom{e_3}{e_2}\binom{e_2}{e_1}\binom{e_1}{e_0}\quad \textit{for } r>0. \tag{2:5}$$

Summing (2:5) *through all* $e\in\binom{I_{1+n}}{1+r}$ *we get the number* $p(n, 1+n)$ *of all* $(1+r)$ — *chains* $\subset P(n)$:

$$p(n, 1+r)=\sum_{e_r=r}^{n}\binom{n}{e_r}\sum_{e_{r-1}=r-1}^{e_r-1}\binom{e_r}{e_{r-1}}\cdots\sum_{e_2=2}^{e_3-1}\binom{e_3}{e_2}\sum_{e_1=1}^{e_2-1}\binom{e_2}{e_1}\sum_{e_0=0}^{e_1-1}\binom{e_1}{e_0}. \tag{2:6}$$

In particular

$$p(n, 1)=2^n, \tag{2:7}$$

$$p(n, 1+n)=n! \tag{2:8}$$

2:9. We put also

$p(n, 0)=1$ since the empty set v is considered as a chain in $(P(n), \subset)$ for every $n\in N$. Thus v is a part as well as a member of $P(n)$ for every $n\in N$.

Proof. The particular case $r=0$ yields $e=(e_0)$ and the corresponding summation in (2:6) becomes $\sum_{e_0=0}^{n}\binom{n}{e_0}=2^n$, i. e. (2:7) is holding. Let us consider the case that $r>1$. Since $kx_r=e_r$, x_r is any member of $\binom{I_r}{e_r}$ and thus x_n can assume $\binom{n}{e_r}$ values. Since $x_{r-1}\subset\neq x_r$; $kx_{r-1}=e_{r-1}$, x_{r-1} is any $\in\binom{x_r}{e_{r-1}}$, thus x_{r-1} assumes $\binom{e_r}{e_{r-1}}$ values, ... By induction argument we infer that the formula (2:6) holds. In particular,

$$p(n, 1+n)=\binom{n}{n}\binom{n}{n-1}\binom{n-1}{n-2}\cdots\binom{3}{2}\binom{2}{1}\binom{1}{0}=1\cdot n(n-1)\ldots 2\cdot 1=n!$$

Taking the sum of the numbers $p(n, 1+r)$ for $r=0, 1, \ldots, n$ we obtain the following:

2:10. Theorem. *The number of non empty chains in* $(P(n), \subset)$ *equals*

$$L_n = \sum_{r=0}^{n} p(n, 1+r) = \sum_{r=0}^{n} \sum_{e} \binom{n}{e_r}\binom{e_r}{e_{r-1}} \cdots \binom{e_2}{e_1}\binom{e_1}{e_0},$$

$$e := (e_0, e_1, \ldots, e_r) \in \binom{I_{1+n}}{1+r}.$$

2:11. Theorem. *For every natural number n and every* $a \in \{0, 1, \ldots, n\}$ *one has*

$$(2:12) \quad p(n, 1+a) = \sum_{e_a=a}^{n} \binom{n}{e_a} \left(\sum_{e_{a-1}=a-1}^{e_a-1} \binom{e_a}{e_{a-1}} \left(\sum_{e_{a-2}=a-2}^{e_{a-1}-1} \binom{e_{a-1}}{e_{a-2}} \left(\cdots \right.\right.\right._{a-3}$$

$$\cdots \left(\sum_{e_1=1}^{e_2-1} \binom{e_2}{e_1} \left(\sum_{e_0=0}^{e_1-1} \binom{e_1}{e_0} \right)_0 \right)_1 \cdots \Big)_{a-2} \Big)_{a-1} = \Delta^a 2^n =$$

$$= \sum_{s=0}^{a} (-1)^s \binom{a}{s} (a+2-s)^n.$$

Proof. The theorem was proved for $a=0$; (v. 1:1, 2:7). Let $a \geq 1$ and assume that (2:12) holds for every number a of indicated „parentheses"; let us prove it also for $1+a$ parentheses, i. e. that (2:12) holds.

Let us consider $(2:12)_2$; on applying 1:10 L step by step to expressions $_0(\ )_0$, $_1(\ )_1$, $\ldots$, $_{a-2}(\ )_{a-2}$ the second part $(2:12)_2$ of (2:12) becomes

$$(2:12)_2 \quad = \sum_{e_a=a}^{n} \binom{n}{e_a} \Delta^a 1^{e_a}; \text{ further, this equals } \left(\text{in virtue of } \sum_{n}^{a} = \sum_{0}^{n} - \sum_{0}^{a-1}\right)$$

$$\sum_{e_a=0}^{n} \binom{n}{e_a} 1^{e_r} - \sum_{e_a=0}^{r-1} \binom{n}{e_a} \Delta^a 1^{e_a} =$$

(apply 1:10 L to the first sum and 1:9 Th to the second sum)

$$= \Delta^{a\,2n} - 0 = (2:12)_3$$

As to the equality $(2:12)_3 = (2:12)_4$ see (1:4) for $(n, m, h) = (m, n, 1)$. Q.E.D.

2:13. Remark. The equality $(2:12)_1 = (2:12)_4$ is due to M. Popadić [1951 formula (2)].

3. Another expression for $p(n, a)$. Let us reconsider the relation

$$(3:1) \qquad p(n, a) := \sum_{e} n(e),\ n(e) := \binom{e_1}{e_0}\binom{e_2}{e_1} \cdots \binom{n}{e_{a-1}} =$$

$$= \frac{n!}{e_0!\,(e_1-e_0)!\,(e_2-e_1)! \cdots (e_{a-1}-e_{a-2})!\,(n-e_{n-1})!},$$

$$e := (e_0, e_1, \ldots, e_{a-1}) \in \binom{I_{1+n}}{a}.$$

Instead of summing in (3:1) over $e\in\binom{I(1+n)}{a}$ we shall sum over e_0 and over the sequence d of differences

(3:2) $d_0=e_1-e_0,\ d_1=e_2-e_1,\dots,\ d_{a-2}=e_{a-1}-e_{a-2}$ (the number of terms of this sequence is $a-1$). We have

(3:3) $e_1=e_0+d_0,\ e_2=e_0+d_0+d_1,\dots,e_{a-1}=e_0+d_0+d_1+\cdots+d_{a-2}=e_0+sd,$ where

(3:4) $$sd:=d_0+d_1+\cdots+d_{a-2}.$$

Since $e_{a-1}\leqslant n$ we infer that for a given sequence (3:2) of differences the greatest admissible value of e_0 satisfies $n=e_0+d_0+d_1+\cdots+d_{a-2}$, i. e.

(3:5) $$e_0\in\{0,1,\dots,n-sd\}.$$

3:6. Lemma. *Let $d=d_0,d_1,\dots,d_{a-2}$ be any sequence of positive integers such that $sd\leqslant n$.*

If $d'=d_0',\dots,d_{a-2}'$ is any permutation of the sequence d, then

(3:7) $$n[d]=n[d'],$$

where, by definition, $n[d]$ denotes $\sum_e n(e)$, e satisfying (3:1), (3:2), in other words,

(3:8) $$n[d]=\sum_{e_0=0}^{n-sd}\frac{n!}{e_0!\,d_0!\,d_1!\dots d_{a-2}!(n-e_{a-1})}.$$

As a matter of fact, the last expression yields

$$n[d]=\frac{(n+1-sd)(n+2-sd)\dots n}{d_0!\,d_1!\dots d_{a-2}!}\sum_{e_0=0}^{n-sd}\binom{n-sd}{e_0}\quad\text{because } e_{a-1}=e_0+sd,$$

$$\frac{n!}{e_0!(n-sd-e_0)!}=\binom{n-sd}{e_0}(n+1-sd)(n+2-sd)\dots(n-1)n.$$

Thus

(3:9) $$n[d]=\frac{(n+1-sd)(n+2-sd)\dots n}{d_0!\,d_1!\dots d_{a-2}!}2^{n-sd}.$$

By the same argument we find the same expression for $n[d']$: $n[d']=(3:9)_2$. This means that (3:7) is holding.

3:10. Main theorem. *For any given 2-un (n,a) of natural numbers, let $p(n,a)$ denote the cardinal number of the system of all chains in $(P(n),\subset)$, each of cardinality a; then*

(3:11) $$p(n,a)=\sum_d d!\,\frac{(n+1-sd)(n+2-sd)\dots n}{d_0!\,d_1!\dots d_{a-2}!}2^{n-sd}=\Delta^{a-1}2^n,$$

where $d:=(d_0,d_1,\dots,d_{a-2})$ runs through the set of all increasing sequences

(3:12) $$d\dots d_0\leqslant d_1\leqslant\dots\leqslant d_{a-2}$$

of natural numbers satisfying

(3:13) *$sd := d_0 + d_1 + \cdots + d_{a-2} \leqslant n$; in particular $d_0 \geqslant 1$; $d!$ denotes the number of all permutations of the sequence d.*[1]

In particular

(3:14) $$p(n, n+1) = n! = \Delta^n 2^n$$

(3:15) $$p(n, n) \quad = n! \, \frac{n+3}{2} = \Delta^{n-1} 2^n.$$

(3:16) $$p(n, n-1) = n! \left(\frac{n^2}{8} + \frac{13}{24} n + \frac{5}{12}\right) = \Delta^{n-2} 2^n.$$

(3:17) $$p(n, n-2) = n! \left(\frac{1}{48} n^3 + \frac{1}{12} n^2 + \frac{1}{48} n - \frac{1}{24}\right) = \Delta^{n-3} 2^n.$$

Proof of the theorem.

3:18. First of all, instead to perform the summation in (3:1) for $p(n, a)$ over $e \in \binom{I(n+1)}{a}$ we shall do the summation over e_0 and over sequences (3:12); from (3:12) and (3:13) we infer that there is a one-to-one correspondence between e's and e_0, d's. The formula (3:1) yields

$$p(n, a) = \sum_{e_0, d} \frac{n!}{e_0! \, d_0! \, d_1! \ldots d_{a-2}! \, (n - e_0 - sd)!} =$$

$$= \sum_d \sum_{e_0=0}^{n-sd} \frac{n!}{e_0! \, d_0! \ldots d_{a-2}! \, (n - sd)!} = \sum_d n[d].$$

Here d means any sequence $d = d_0, d_1, \ldots, d_{a-2}$ of positive integers such that $sd \leqslant n$. If d' is the normal permutation of d, i. e. such one that $d_0' \leqslant d_1' \leqslant \cdots \leqslant d_{a-2}'$, then by virtue of the Lemma 3:6 we have $n[d] = n[d']$; consequently, $\sum n[d] = \sum [d']$, and this is exactly the content of the requested relation $(3:11)_1 = (3:11)_2$.

3:19. Case $p(n+1)$, i. e. $e = (0, 1, \ldots, n)$, $d = 1, 1, \ldots, 1$, $sd = n$; the formula (3:11) becomes precisely (3:14).

3:20. Case $p(n, n)$. The conditions (3:12), (3:13), $a = n$ yield that $sd = n-1$ or $sd = n$. If $sd = n-1$, then $d_i = 1$ $(i = 0, 1, \ldots, n-2)$; $d! = 1$; the corresponding part in $p(n, n)$, according to (3:11), equals

$$1 \cdot \frac{2 \cdot 3 \ldots n}{1! \, 1! \ldots} 2 = 2n!$$

[1] If $f = f_1, f_2, \ldots$ is any sequence of objects, a permutation of f is any sequence $f' = f_1', f_2', \ldots$ such that the frequency νf_k of every object f_k in f equals the frequency of the same object of f', and that the frequency $\nu f_k'$ of every term f_k' of f' equals the frequency of the same object in f.

If $f!$ denotes the total number of permutations of $f := (f_1 f_2, \ldots, f_n)$ then one knows that $f = n! : \prod_x (\nu x)!$, $x \in \{f_1, f_2, \ldots, f_n\}$.

If $sd=n$, then necessarily $d=(1)_{n-2}$, 2; thus $d!=\frac{(n-1)!}{(n-2)!}=n-1$ and the corresponding part in $p(n,n)$ is $(n-1)!\ \frac{n!}{2!}\ 2^0=\frac{n-1}{2!}n!$. Consequently,

$$p(n,n)=2\,n!+\frac{n-1}{2}\,n!=n!\ \ \frac{n+3}{2}=(3:15)_2.$$

3:21. Expression for $p(n,n-1)=\Delta^{n-2}\,2^n$.

If we have $a=n-1$, then in (3:11) the sequences d are of length $a-1=$ $=n-2$ thus we have $1\leqslant d_0\leqslant d_1\leqslant\cdots\leqslant d_{n-3}$ and $sd:=d_0+d_1+\cdots+d_{n-3}\leqslant n$; therefore $sd\in\{n-2,\ n-1,\ n\}$.

(1) If $sd=n-2$, then necessarily $d=(1)_{n-2}:=(\underbrace{1,1,\ldots,1}_{n-2})$; then $d!=1$, $n+1-sd=3$, and the term under $\sum$ in (3:11) reads $\frac{3\ 4\ldots n}{1!\ 1!\ 1!\ldots 1!}\ 2^2=2\,n!$

(2) If $sd=n-1$, then $d=(1)_{n-3}\,2$, $d!=\frac{(n-2)!}{(n-3)!\ 1!}=n-2$; $n+1-sd=2$; according to (3:11) we have

$$n[d]=(n-2)\ \ \frac{n!}{2!}\,2^1=(n-2)\,n!$$

(3) If $sd=n$, then $d=(1)_{n-3}\,3$ or $d=(1)_{n-4}\,2{,}2$.

(3:1) The case $d=(1)_{n-3}$, 3 yields $d!=\frac{(n-2)!}{(n-3)!}=n-2$, $n+1-sd=1$, $n[d]=$ $=(n-2)\frac{n!}{3!}\ \ 2^0=(n-2)\frac{n!}{3!}$.

(3:2) The case $d=(1)_{n-4}(2)_2$ yields

$$d!=\frac{(n-2)!}{(n-4)!\ 2!}=\frac{(n-3)(n-2)}{2},\quad n+1-sd=1,$$

$$n[d]=\frac{(n-3)(n-2)}{2}\ \ \frac{n!}{2!\,2!}\ \ 2^0=\frac{(n-3)(n-2)}{8}\,n!.$$

(4) The summation of all these 4 cases yields

$$p(n,n-1)=2\,n!+(n-2)\,n!+(n-2)\,\frac{n!}{3!}+\frac{(n-3)(n-2)}{8}\,n!=$$

$$=n!\left(\frac{n^2}{8}+\frac{24+4-15}{24}n+\frac{48-48-8+18}{24}\right)=n!\left(\frac{n^2}{8}+\frac{13}{24}n+\frac{5}{12}\right)=(3:16)_2.$$

3:22. Number $p(n, n-2) = \Delta^{n-3} 2^n$. We shall apply the main theorem 3:10 putting $a = n-2$; consequently, for any $e \in \binom{I(n+1)}{a}$ the corresponding difference sequence d has $n-3$ terms: $d = d_0 \leqslant d_1 \leqslant \cdots \leqslant d_{n-4}$. Therefore we assume $n > 4$. Since $n-3 \leqslant sd \leqslant n$ we have to consider the following 4 cases:

(1) $sd - n - 3$; in this case: $d = (1)_{n-3}$, $d! = 1$, $n+1-sd = 4$, therefore $n[d] = \frac{n!}{3!} 2^3$;

(2) $sd = n-2$; in this case $d = (1)_{n-4}, 2$; $d! = \frac{(n-3)!}{(n-4)!} = n-3$; $n+1-sd = 3$; therefore the summand in $(3:11)_2$ becomes

$$n[d] = (n-3) \frac{1}{2} \cdot \frac{n!}{2!} 2^2 = n!\,(n-3);$$

(3) $sd = n-1$; in this case $d \in ((1)_{n-4}, 3)$; $((1)_{n-5} (2)_2)$, $n+1-sd = 2$;

(3.1) If $d = (1)_{n-4}\, 3$, then $d! = n-3$ and

$$n[d] = (n-3) \frac{n!}{3!} 2;$$

(3.2) If $d = (1)_{n-5} (2)_2$, then $d! = \frac{(n-3)!}{(n-5)!\, 2!} = \frac{(n-4)(n-3)}{2}$, and

$$n[d] = \frac{(n-4)(n-3)}{2} \frac{n!}{2!\, 2!} 2^1;$$

(4) $sd = n$; in this case $d = (1)_{n-4}\, 4$ or $d = (1)_{n-5}\, 2 \cdot 3$, or $d = (1)_{n-6} (2)_3$, $n+1-sd = 2$.

(4.1) If $d = (1)_{n-4}\, 4$, then $d! = n-3$,

$$n[d] = (n-3) \frac{n!}{4!} 2^0;$$

(4.2) If $d = (1)_{n-5}\, 2{,}3$ then $d! = \frac{(n-3)!}{(n-5)!} = (n-4)(n-3)$,

$$n[d] = (n-4)(n-3) \frac{n!}{2!\, 3!} 2^0.$$

(4.3) If $d = (1)_{n-6} (2)_3$, then $d! = \frac{(n-3)!}{(n-6)!\, 3!} = \frac{(n-5)(n-4)(n-3)}{3!}$,

$$n[d] = \frac{(n-5)(n-4)(n-3)}{3!} \frac{n!}{(2!)^3} 2^0.$$

Summing the obtained values for $n[d]$ one gets $(3:17)_2$.

4. Table and some properties of the numbers $p(n, s)$.

4:1. *Table of numbers* $p(n, r) = \Delta^{r-1} 2^n$.

By either of the formulae (2:6), (2:11), (3:10) one could establish the following values for $\Delta^r 2^n$ $(= p(n, 1+r))$.

$n \backslash r$	0	1	2	3	4	5	6	7	8
1	2	1							
2	4	**5**	2						
3	8	**19**	18	6					
4	16	65	**110**	84	24				
5	32	211	570	**750**	480	120			
6	64	665	2702	5460	**5880**	3240	720		
7	128	2059	12138	35406	**57120**	52080	25200	5040	

Remark. Maximal members in every line are bold-faced.

4:2. We observe and check that every row of the table has an initial strictly increasing segment and a terminating one which is strictly decreasing.

4:2:1. Is the union of these maximal parts the row itself? In other words, is every row of $\Delta^r 2^n$ (n is fixed) decomposable into an initial maximal segment which is strictly increasing and the remaining terminal segment which is strictly decreasing?

4:2:2. If q_n is the ratio of the lengths of these two segments, find $\underline{\lim}\, q_n$, $\overline{\lim}\, q_n$.

4:2:3. E. g. one checks easily that

$$6 < n \in N \Rightarrow p(n, n-2) > p(n, n-1) > p(n, n) > p(n, n+1).$$

As a matter of fact, for the expression

$$g(n) := n!^{-1}(p(n, n-2) - p(n, n-1))$$

in virtue of (3:16), (3:17) we have

$$g(n) = \frac{1}{48} n^3 - \frac{1}{24} n^2 - \frac{25}{48} n - \frac{11}{24}$$

and one sees that $g(n) > 0$ for $6 < n \in N$.

4:3. Function $2(n)$. For $n \in N$ let $2(n)$ be defined by

(4:3:1) $$\Delta^{2(n)} 2^n = \sup_k \Delta^k 2^n;$$

4:4:2. We guess that $2(n) \geqslant \left[\frac{1+n}{2}\right]$ for every $n \in N \backslash \{3\}$.

4:4. More generally, for any $r \in \{0, 1, 2, \ldots\}$ and any natural number n let $r(n)$ be the first natural number $x \in N$ such that

(4:4:1) $$\Delta^x r^n = \sup_k \Delta^k r^n.$$

4:4:2. **Problem.** Is $r(n) \geqslant \left[\frac{1+n}{2}\right]$ for every $n \in N \setminus \{3\}$?

4:4:3. **Problem.** Is $r(n)$ the unique solution of the relation (4:4:1)?

4:4:4. **Problem.** Do the relations $1 \neq \Delta^a 2^n = \Delta^{a'} 2^{n'}$ have only the trivial solution $(a, n) = (a', n')$?

4:4:5. **Problem.** Find the solution set of $\Delta^a b^c = \Delta^{a'} b^{c'} \neq 1$.

Remark that e. g. for the case $b = b' = 0$ one has $\Delta^2 0^3 = \Delta^3 0^3$ $(=6)$. Is it the unique non trivial solution of $\Delta^a 0^c = \Delta^{a'} 0^{c'}$?

BIBLIOGRAPHY

Felgner, Ulrich [1971] *Models of ZF-Set Theory*. Lecture Notes 223 (1971), VI + 173.

Kurepa, Đuro [1953:1] *Über das Auswahlaxiom*. Mathematische Annalen, 126 (1953), pp. 381—384.

[1957:1] *Partitivni skupovi i uređeni lanci — Partitive sets and ordered chains* (Rad Jugosl. Akademije Znanosti i umjetnosti, knjiga 302, Odjel za mat. fiz. i tehn. nauke, knjiga 6 (Zagreb, 1957) 197—203—205—235.

Kurepa, Đuro [1978:4] *A combinatorial interpretation of a sequence of integers*. pp. 1179—1184. Colloquia Math. Soc. János Bolyai 18, Combinatorics, Keszthely (Hungary), 1976 (Budapest 1978), North Holland. Publ. Comp. Edited by A. Hajnal and Vera T. Sós.

Popadić, Milan S. [1951] *O broju lanaca u konačnim partitivnim množinama. — On the number of chains in a kind of ordered finite sets*. Annuaire Fac. Philosophie Univ. Skopje, Section Sci. natur. 4 (Skopje, 1951) № 4, pp. 4—8—10.

Sierpiński, Wacław [1922] *Sur un problème concernant les sous-ensemles croissants du continu*. Fund. Math. 3 (Warszawa 1922) 109—112. *Oeuvres choisies* II (Warszawa 1975) 444—447.

Matematički institut
Knez Mihailova 35/I
11000 Beograd
Jugoslavija