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ON 0-MINIMAL (0,2)-BI-IDEALS OF SEMIGROUPS

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The notion of (m, n)-ideals of semigroups was introduced by S. Lajos [3]. In this note we describe (0,2)-ideals, (1,2)-ideals and 0-minimal (0,2)-ideals. Also, we introduce the notion of (0,2)-bi-ideals and 0-(0,2)-bisimple semigroups, we describe 0-minimal (0,2)-bi-idelas and prove that a semigroup $S=S^0$ is 0-(0,2)-bi-simple if and only if S is left 0-simple (Theorem 3).

A subsemigroup A of a semigroup S is (0,2)-ideal of S if $SA^2 \subseteq A$.

Let A be a (0,2)-ideal of a semigroup S. Then we have $(A \cup SA)A = A^2 \cup SA^2 \subseteq A$ so that A is a left ideal of the left ideal $A \cup SA$ of S. Conversely, let L be a left ideal of S and A be a left ideal of L. Then, $SA^2 \subseteq SLA \subseteq LA$ so that A is a (0,2)-ideal of S. We have therefore proved

Lemma 1. *If A is a subset of a semigroup S, then the following statements are equivalent*:

(i) *A is a* (0,2)*-ideal of S.*

(ii) *A is a left ideal of some left ideal of S.*

A subsemigroup A of a semigroup S is a (1,2)-ideal of S if $ASA^2 \subseteq A$.

Theorem 1. *Let A be a subset of a semigroup S. The following statements are equivalent.*

(i) *A is a* (1,2)*-ideal of S.*

(ii) *A is a left ideal of some bi-ideal of S.*

(iii) *A is a bi-ideal of some left ideal of S.*

(iv) *A is a* (0,2)*-ideal of some right ideal of S.*

(v) *A is a right ideal of some* (0,2)-ideal of S.

Proof. (i) $\Rightarrow$ (ii). Let A be a (1,2)-ideal of S, i.e., A is a subsemigroup of S and $ASA^2 \subseteq A$. Then $(A \cup ASA)A = A^2 \cup ASA^2 \subseteq A$, so that A is a left ideal of the bi-ideal $A \cup ASA$ of S.

(ii) $\Rightarrow$ (iii). Left A be a left ideal of some bi-ideal B of S, i.e.,

$A \subseteq B$, $BA \subseteq A$ and $BS^1B \subseteq B$, so we have

$A(A \cup SA)A = A^3 \cup ASA^2 \subseteq A \cup BSBA \subseteq A \cup BA = A.$

Therefore A is a bi-ideal of the left ideal $A \cup SA$ of S.

(iii) $\Rightarrow$ (iv). Left A be a bi-ideal of some left ideal L of S, i.e.,

$A \subseteq L$, $AL^1A \subseteq A$ and $SL \subseteq L$. Then

$(A \cup AS)A^2 = A^3 \cup ASA^2 \subseteq A \cup ASLA \subseteq A \cup ALA = A$

so that A is a (0,2)-ideal of the right ideal $A \cup AS$ of S.

(iv) $\Rightarrow$ (v). Let A be a (0,2)-ideal of some right ideal R of S, i.e.,

$A \subseteq R$, $RA^2 \subseteq A$ and $RS \subseteq R$. Then

$A(A \cup SA^2) = A^2 \cup ASA^2 \subseteq A \cup RSA^2 \subseteq A \cup RA^2 = A$.

Therefore, A is a right ideal of the (0,2)-ideal $A \cup SA^2$ of S.

(v) $\Rightarrow$ (i). Let A be a right ideal of a (0,2)-ideal R of S, i.e.

$A \subseteq R$, $AR \subseteq A$ and $SR^2 \subseteq R$. Then $ASA^2 \subseteq ASR^2 \subseteq AR \subseteq A$. Therefore, A is a bi-ideal of S.

Lemma 2. *A subsemigroup A of a semigroup S is a* (1,2)-*ideal of S if and only if there exist a* (0,2)-*ideal L of S and a right ideal R of S such that* $RL^2 \subseteq A \subseteq R \cap L$.

Proof. Let A be a (1,2)-ideal of S. Let $L = A \cup SA^2$ and $R = A \cup AS$. Then $RL^2 = (A \cup AS)(A \cup SA^2)^2 = (A^2 \cup ASA)(A \cup SA^2) = A^3 \cup ASA^2 \subseteq A$. Hence, $RL^2 \subseteq A \subseteq R \cap L$. Conversely, let R be a right ideal of S and let L be a (0,2)-ideal of S so that $RL^2 \subseteq A \subseteq R \cap L$. Then

$ASA^2 \subseteq (R \cap L)S(R \cap L)(R \cap L) \subseteq RSL^2 \subseteq RL^2 \subseteq A$,

so that A is a (1,2)-ideal of S.

Let $S = S^0$. Every left ideal of S is a (0,2)-ideal of S. Let L be a 0-minimal (0,2)-ideal of S. If A is a left ideal of S, contained in L, then $A = \{0\}$ or $A = L$. It is natural to ask: What can we say on a (0,2)-ideal of S, contained in some 0-minimal left ideal of S?

Lemma 3. *Let L be a* 0-*minimal left ideal of $S = S^0$ and A a sub-semigroup of L. Then A is a* (0,2)-*ideal of S if and only if $A^2 = \{0\}$ or $A = L$.*

Proof. Let A be a (0,2)-ideal of S contained in L. As SA^2 is a left ideal of S and $SA^2 \subseteq A \subseteq L$, we have $SA^2 = \{0\}$ or $SA^2 = L$. If $SA^2 = L$, then $A = L$. Let $SA^2 = \{0\}$. Then A^2 is a left ideal of S, contained in L, so we have $A^2 = \{0\}$ or $A^2 = L$. If $A^2 = L$, then $A = L$. Hence $A^2 = \{0\}$ or $A = L$. Converse is trivial.

The following lemma gives a description of 0-minimal (0,2)-ideals of a semigroup S (see [1] and [2] for 0-minimal ideals and 0-minimal bi-ideals).

Lemma 4. *Let L be a* 0-*minimal* (0,2)-*ideal of a semigroup $S = S^0$. Then $L^2 = \{0\}$ or L is a* 0-*minimal left ideal of S.*

Proof. Since $L^2 \subseteq L$ and $S(L^2)^2 = SL^2L^2 \subseteq LL = L^2$ we have that L^2 is (0,2)-ideal of S, contained in L. Then $L^2 = \{0\}$ or $L^2 = L$. But, $L^2 = L$ implies $SL \subseteq L$ so L is a left ideal of S and clearly 0-minimal. Therefore, $L^2 = \{0\}$ or L is a 0-minimal left ideal of S.

Corollary 1. *Let S be a semigroup without zero. Then L is a minimal (0,2)-ideal of S if and only if L is a minimal left ideal of S.*

Let S be a semigroup without zero and A be a minimal (2,1)-ideal of S. Then $A^2SA \subseteq A$. Since A^2SA is a (2,1)-ideal of S we have $A^2SA = A$, so that $ASA = A^2SASA \subseteq A^2SA = A$ i.e. A is a minimal bi-ideal of S. Conversely, let A be a minimal bi-ideal of S. Let $B \subseteq A$ and $B^2SB \subseteq B$. Since B^2SB is a bi-ideal of S we have $B^2SB = A$, i.e. $A \subseteq B$. Therefore $A = B$.

So we have proved

Lemma 5. *Let S be a semigroup without zero and let A be a nonempty subset of S. The following statements are equivalent.*

(i) *A is a minimal (2,1)-ideal of S.*

(ii) *A is a minimal bi-ideal of S.*

Definition 1. A subsemigroup A of a semigroup S is called a (0,2)-*bi-ideal* of S if A is a bi-ideal of S and also a (0,2)-ideal of S. A (0,2)-bi-ideal A of S is called 0-*minimal* if (i) $A \neq \{0\}$ and (ii) $\{0\}$ is the only (0,2)-bi-ideal of S properly contained in A. A semigroup $S = S^0$ is called 0-(0,2)-*bisimple* if (i) $S^2 \neq \{0\}$ and $\{0\}$ is the only proper (0,2)-bi-ideal of S.

It is easy to seet that $\{a\} \cup \{a^2\} \cup aSa \cup Sa^2 \cup Sa^2Sa$ is the principal (0,2)-bi--ideal of S generated by the element a of S.

Lemma 6. *Let A be a subset of a semigroup S. The following statements are equivalent.*

(i) *A is a (0,2)-bi-ideal of S.*

(ii) *A is an ideal of some left ideal of S.*

Proof. (i) $\Rightarrow$ (ii). Let L be a (0,2)-bi-ideal of S i.e. $ASA \subseteq A$ and $SA^2 \subseteq A$. Then we have $A(A \cup SA) = A^2 \cup ASA \subseteq A$ and $(A \cup SA)A = A^2 \cup SA^2 \subseteq A$. Therefore A is an ideal of the left ideal $A \cup SA$ of S.

(ii) $\Rightarrow$ (i). Let A be an ideal of some left ideal L of S. According to Lemma 1, A is a (0,2)-ideal of S and also A is a bi-ideal of S [4].

Theorem 2. *Let A be a 0-minimal (0,2)-bi-ideal of a semigroup S. Then exactly one of the following cases occurs.*

(i) $A = \{0, a\}$, $aS^1a = \{0\}$.

(ii) $A = \{0, a\}$, $a^2 = 0$, $aSa = A$.

(iii) $(\forall a \in A \setminus \{0\})$ $Sa^2 = A$.

P r o o f. Let $a \in A \setminus \{0\}$. Then $Sa^2 \subseteq A$ and Sa^2 is a left ideal of S so Sa^2 is a (0,2)-bi-ideal of S. Therefore $Sa^2 = \{0\}$ or $Sa^2 = A$.

Let $Sa^2 = \{0\}$. Since $a^2 \in A$ we have either $a^2 = a$ or $a^2 = 0$ or $a^2 \in A \setminus \{0\}$. If $a^2 = a$ then $a^3 = aa^2 = a$. It is impossible because $a^3 \in Sa^2 = \{0\}$. Let $a^2 \in A \setminus \{0, a\}$. Then $S^1 \{0, a^2\}^2 = \{0\}$ and $\{0, a^2\} S \{0, a^2\} = a^2 Sa^2 = \{0\}$. Therefore $\{0, a^2\}$ is (0,2)-bi-ideal of S contained in A and $\{0, a^2\} \neq \{0\}$, $\{0, a^2\} \neq A$. This is also impossible because A is 0-minimal (0,2)-bi-ideal of S. Therefore $a^2 = \{0\}$ and $A = \{0, a\}$. The subset aSa is a bi-ideal of S and $S(aSa)^2 = SaSa^2Sa = \{0\}$ so that aSa is (0,2)-bi-ideal of S contained in A. Then $aSa = \{0\}$ or $aSa = A$. Therefore, $Sa^2 = \{0\}$ implies either $A = \{0, a\}$ and $aS^1a = \{0\}$ or $A = \{0, a\}$, $a^2 = 0$ and $aSa = A$.

If $Sa^2 \neq \{0\}$ then $Sa^2 = A$.

According to Theorem 2 and [2] we see that a 0-minimal (0,2)-bi-ideal is either 0-minimal bi-ideal of S (degenerate or nondegenerate) or a nondegenerate 0-minimal left ideal of S.

C o r o l l a r y 2. *Let A be a 0-minimal (0,2)-bi-ideal of S so that $A^2 \neq \{0\}$. Then $A = Sa^2$ for every $a \in A \setminus \{0\}$.*

C o r o l l a r y 3. *A semigroup $S = S^\circ$ is 0-(0,2)-bisimple if and only if $Sa^2 = S$ for every $a \in S \setminus \{0\}$.*

P r o o f. If S is 0-(0,2)-bisimple, then $S^2 \neq \{0\}$ and S is 0-minimal (0,2)-bi-ideal. According to Corollary 2, $S = Sa^2$ for every $a \in S \setminus \{0\}$. Conversely, let $S = Sa^2$ for every element $a \in S \setminus \{0\}$ and let A be a non-null (0,2)-bi-ideal of S. Let $a \in A \setminus \{0\}$. Then $S = Sa^2 \subseteq SA^2 \subseteq A$ so that $S = A$. Since $S = Sa^2 \subseteq S^2$ we have $S = S^2$. Therefore, S is 0-(0,2)-bi-simple.

T h e o r e m 3. *A semigroup $S = S^0$ is 0-(0,2)-bisimple if and only if S is, left 0-simple.*

P r o o f. Every left ideal of a semigroup S is a (0,2)-bi-ideal of S. So if S is 0-(0,2)-bisimple then S is left 0-simple. Conversely, if S is left 0-simple then $Sa = S$ for every $a \in S \setminus \{0\}$ from which it follows that $Sa^2 = Saa = Sa = S$. Hence, by Corollary 3, S is 0-(0,2)-bisimple.

T h e o r e m 4. *Let A be a 0-minimal (0,2)-bi-ideal of S. Then either $A^2 = \{0\}$ or A is left 0-simple.*

P r o o f. Let $A^2 \neq \{0\}$. Then, according to Corollary 2, $Sa^2 = A$ for every $a \in A \setminus \{0\}$. Then $a^2 \in A \setminus \{0\}$ for every $a \in A \setminus \{0\}$ so $a^4 = (a^2)^2 \in A \setminus \{0\}$ for every $a \in A \setminus \{0\}$. Let $a \in A \setminus \{0\}$. Since $Aa^2S^1Aa^2 \subseteq AAa^2 \subseteq Aa^2$ and $S(Aa^2)^2 = SAa^2Aa^2 \subseteq SA^2a^2 \subseteq Aa^2$, Aa^2 is a (0,2)-bi-ideal of S contained in A. Then $Aa^2 = \{0\}$ or $Aa^2 = A$. Since $a^4 \in Aa^2$ and $a^4 \in A \setminus \{0\}$ we have $Aa^2 = A$. According to Corollary 3 and Theorem 3, A is left 0-simple.

The following example shows that there exists a 0-minimal left ideal of the semigroup $S=\{e, f, g, a, 0\}$ which is not a 0-minimal (0,2)-bi-ideal of S.

	e	f	g	a	0
e	e	a	e	a	0
f	0	f	g	0	0
g	g	f	g	f	0
a	0	a	e	0	0
0	0	0	0	0	0

$L=\{0, f, a\}$ is a 0-minimal left ideal of S.

Let $A=\{0, a\}$. Then

$ASA=\{0, a\} S \{0, a\}=\{0, a, e\} \{0, a\}=\{0, a\}=A$

$A^2=\{0\}$ and $SA^2=\{0\}$.

Therefore, A is (0,2)-bi-ideal of S from which it follows that L is not a 0-minimal (0,2)-bi-ideal of S.

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