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# LINEARIZATION OF NONLINEAR DIFEERENTIAL EQUATIONS: THIRD ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS EQUIVALENT TO LINEAR SECOND ORDER EQUATIONS

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**0.** The linearization of nonlinear differential equations has been the topic of numerous investigations [1]—[8].

Inspired by the Pinney's note [1], J. M. Thomas [2] posed the following question: Which equations of order $n$ have general solutions expressible in the form $F(u_1, \ldots, u_n)$, where $u_1, \ldots, u_n$ constitute a variable set of solutions of a linear equation. Only for the first and second order equations complete answers to this question are given ([2]—[5]). Beside the investigations of nonlinear equations equivalent to the linear ones of the same order, the nonlinear equations which, by suitable transformations, are reduced to the linear ones of different order are also studied: Riccati's equations of the first and second order, Painlevé's results [7], [8], etc. All results, in fact, provide a partial reply to the question: Which nonlinear equations of order $n$ have the solution $F(u_1, \ldots, u_m)$, where $u_1, \ldots, u_m$ are linearly independent solutions of a linear equation of order $m$?

This paper will deal with the nonlinear equations of third order equivalent to a linear homogenous equation of second order.

In further text $S(E)$ denotes the set of all solutions of an equation $(E)$.

We suppose that all functions have continuous derivatives of the third order.

**1.** Theorem 1. *Implication*

(I) $$u, v \in S(E_1) \Rightarrow y = F(u, v) \in S(E_2)$$

*where*

$(E_1)$ $$U'' = p(x)U$$

$(E_2)$ $$y''' + h(y'', y', y, p, p') = 0$$

*holds, if and only if one of the following conditions is satisfied:*

(i) *Function h is of the form*

(1) $$h(y'', y', y, p, p') = a(y)y''y' + b(y)y'^3 - 4py' - p'c(y) + c(y)(k(d(y)y'' + e(y)y'^2 - p))^{3/2},$$

*where a, b, c, d, e satisfy the following*

(2) $$cd=1,\ e=d'+d^2,\ a=3(e/d+d),\ b=a'/3+e^2+d^2+2e$$

*and* $k=\text{const}$.

*In this case F is given by*

(3) $$\exp(2\int c(F)\,dF) = (C_1 v^2+C_2 uv+C_3 u^2)\exp(C_4\int (C_1(v/u)^2+C_2 v/u+C_3)^{-1}\,d(v/u)),$$

*where* $C_1$, $C_2$, $C_3$, $C$ *are arbitrary constants such that*

(4) $$C_4^2(4-k^3/4)=k^3(C_1C_3-C_2^2/4);$$

(ii) *Function h has the form*

(5) $$h(y'',y',y,p,p')=-3y''^2/2y'-Q(y)y'^3+2py'.$$

*In this case F is given by*

(6j $$\int (C_1P_1(F)+C_2P_2(F))^{-2}\,dF=v/u+C_3,$$

*where* $C_1, C_2, C_3$ *are arbitrary constants and* $P_1, P_2$ *are independent solutions of the linear equation*:

(7) $$P''(t)=Q(t)P(t)/2.$$

Proof. Suppose that implication (I) is valid, i.e. $u, v\in S(E_1)$, $y=F(u,v)\in S(E_2)$. Substituting $y=F$ into $(E_2)$ we obtain that the following equality must be identically satisfied:

(8) $$\begin{aligned}&p'(F_u u+F_v v)+p(F_u u'+F_v v')+3p(F_{uu}u'u+F_{uv}(u'v+v'u)+F_{vv}v'v)+\\&+F_{uuu}u'^3+3F_{uuv}u'^2v'+3F_{uvv}u'v'^2+F_{vvv}v'^3\\&=h(p(F_u u'+F_v v')+F_{uu}u'^2+2F_{uv}u'v'+F_{vv}v'^2,\ F_u u'+F_v v',\ F, p, p').\end{aligned}$$

Equation (8) is, in fact, a functional equation with unknown functions $h$ and $F$. In further text we will solve that equation.

From (8), after differentiating twice with respect to $p'$, it follows that $h_{p'p'}=0$, and we conclude $h(r,s,t,p,p')=A(r,s,t,p)p'+B(r,s,t,p)$ where $A$, $B$ satisfy:

(9) $$A(p(F_u u+F_v v)+F_{uu}u'^2+2F_{uv}u'v'+F_{vv}v'^2,\ F_u u'+F_v v',\ F, p)=F_u u+F_v v,$$

(10) $$\begin{aligned}&B(p(F_u u+F_v v)+F_{uu}u'^2+2F_{uv}u'v'+F_{vv}v'^2,\ F_u u'+F_v v',\ F, p)\\&=p(F_u u'+F_v v')+3p(F_{uu}u'u+F_{uv}(u'v+v'u)+F_{vv}v'v)\\&+F_{uuu}u'^3+3F_{uuv}u'^2v'+3F_{uvv}u'v'^2+F_{vvv}v'^3.\end{aligned}$$

From (9), we conclude that $A$ depends only on $F$, i.e.

(11) $$A(F)=F_u u+F_v v.$$

Let $A(F)\not\equiv 0$. Then, after substituion $A(F)=T(F)/T'(F)$, we obtain:

$$T(F(u, v))=uG(v/u), \tag{12}$$

where $G$ is an arbitrary function.

Using (12), from (10) we find

$$\begin{aligned} B(r, s, F, p) = & -3(T''/T'+T'/T)rs-(T'''/T'+3T''/T)s^3+4ps \\ & -C(v/u)T((T'r+T''s^2-pT)/T)^{3/2}/T', \end{aligned} \tag{13}$$

where $T=T(F)$, and $C(t)=(G'''(t)G(t)+3G''(t)G'(t))(G''(t)^3G(t))^{-1/2}$.

Since $B$ depends only on $r, s, F, p$, it follows that $C(t)=k^{3/2}=\text{const}$, and $G$ is not arbitrary, but satisfies the equation

$$G'''G+3G''G'=(G(kG'')^3)^{1/2}.$$

The general solution of the above equation (see [9] eq. 7. 12a) is

$$G(t)=(C_1t^2+C_2t+C_3)^{1/2} \exp\ (C_4\int(C_1t^2+C_2t+C_3)^{-1}dt/2),$$

where $C_1, C_2, C_3, C_2$ are arbitrary constants such that (4) holds.

After substitution $a=3(T''/T'+T'/T)$, $b=(T'''/T'+3T''/T)$, $c=T/T'=A$, $d=T'/T$, $e=T''/T$, from the above, we find that $h$ is given by (1), with (2), $F$ by (3), with (4), and the first condition is obtained.

Let $A(F)=0$. Then, from (11), we have

$$F(u, v)=H(v/u), \tag{14}$$

where $H$ is an arbitrary function.

From (10) we get

$$B(r, s, H, p)=-2ps+Q(H)s^3+3r^2/2s, \tag{15}$$

where $Q$ is defined by

$$H'''/H'^3-3H''^2/2H'^4=Q(H). \tag{16}$$

Equation (16), under the transformation $H'=P(H)$, bccomes (4). Then we conclude that $h$ is given by (5) and $F$ is determined by (6), and the first part of the theorem is proved.

The second part of the theorem follows immediately.

**2.** By a similar method we can prove the following result:

Theorem 2. *Implication*

$$u, v\in S(E_3)\Rightarrow y=F(u, v)\in S(E_4), \tag{I'}$$

*where*

$$U''+f(x)U'+g(x)U=0, \tag{E$_3$}$$

$$y'''+h(y'', y', y, f, g, f', g')=0, \tag{E$_4$}$$

*hold, if and only if one of the following conditions is satisfied*:

(i) *Function h is given by*

$$(17)\qquad h(y'',y',y,f,g,f',g')=a(y)y''y'+b(y)y'^3+f(3y''+a(y)y'^2)$$
$$+(f'+2f^2+4g)y'+(g'+2fg)c(y)+c(y)(k(d(y)y''+e(y)y'^2+fc(y)y'+g))^{3/2}$$

*where* $a, b, c, d, e$ *satisfy* (2).

*In this case* $F$ *is determined by* (3), *with* (4).

(ii) $h$ *is of the form*

$$(18)\qquad h(y'',y',y,f',g')=-3y''^2/2y'-Q(y)y'^3+(f'+f^2/2-2g)y'.$$

$F$ *is determined by* (6), *with* (7).

**3.** In this section we give some remarks and examples.

1° If we introduce new unknown function $z$, by $T(z)=T(y)\exp(-\int f(x)dx/2)$, $T(s)=\exp(\int c(s)\,ds)$, then equation $(E_4)$—(17)—(2) becomes an equation of the form $(E_2)$—(1)—(2), where $p=f'/2+f^2/4-g$.

2° Substitution $z=T(y)^2$ reduces equations $(E_2)$—(1)—(2) and $(E_4)$—(17)—(2) to

$$(19)\qquad z'''+2z(k(z''/2z-z'^2/4z^2-p)^{3/2}=4pz'+2p'z$$

$$(20)\qquad z'''+3fz''+(f'+2f+4g)z'+(2g'+4fg)z$$
$$+2z(k(z''/2z-z'^2/4z^2+fz'/z+g)^{3/2}=0,$$

respectively.

3° Transformation $z=T(y)^2\exp(\int(x)\,dx)$ reduces equation $(E_4)-(17)-(2)$ to (19), where $p=f'/2+f^2/4-g$. Equation (19) is a cannonical equation for the equations of the form $(E_4)-(17)-(2)$, i.e. these equations can be obtained from (19) by substitution $z=\eta(y)\xi(x)$

4° Equations of the form $(E_4)$—(18), i.e. $(E_2)$—(5) were treated by *P.* Painlevé [7], [8]. Putting $p=0$ into $(E_2)$—(5), we obtain the result from [7], [8].

5° Taking $k=0$ in (20), we obtain the linear third order equation

$$z'''+3fz''+(f'+2f+4g)z'+(2g'+4fg)z=0,$$

which has the general solution $z=C_1u^2+C^2uv+C_3v^2$ ($u$, $v$ are linearly independent solutions of $(E_3)$, $C_1$, $C_2$, $C_3$ are arbirary constants). This is a well known result (see, for example, [9], eq. 3.26).

6° Equations 7.8. and 7.9. from [9] are the form $(E_2)$—(1)—(2) and the equation 7.10. from [9] is of the form $(E_2)$—(5).

**4.** We shall return to problems on linearization of differential equations in other papers.

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