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ON SOME CLASSES OF LINEAR EQUATIONS, III

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In [1] and [2] we considered the linear equation in x:

$$P(L)x=0 \tag{1}$$

where P is an m-th degree polynomial over C, $x\in V$, V is a commutative algebra over C and L is a linear operator on V.

In the case when L belongs to one of the classes $H(V)$, $K(V)$ or $D_\alpha(V)$, defined in [1], we drew some conclusions regarding the form of the general solution of the equation (1).

In particular, we proved the following theorems (Theorems 5 and 6 from [1]).

T h e o r e m 1. *If $L\in K(V)$ and if $v_1, \ldots, v_m$ are linearly independent solutions of the equation* (1), *then its general solution is*

$$x=\sum_{k=1}^{m} u_k v_k$$

where $u_k\in \ker L$ are arbitrary.

T h e o r e m 2. *Suppose that $\lambda_1, \ldots, \lambda_m$ are distinct roots of P and that they are, at the same time, characteristic values of $L\in K(V)$. If $v_1, \ldots, v_m$ are the corresponding characteristic vectors, then*

$$x=\sum_{k=1}^{m} u_k v_k \tag{2}$$

where $u_1, \ldots, u_m\in \ker L$ are arbitrary, is the general solution of (1).

Since $D_\alpha(V)\subset K(V)$ (see Theorem 10 of [1]), Theorems 1 and 2 also apply to operators from $D_\alpha(V)$.

In this note we shall give the form of the general solution of (1) in the case when $L\in D_\alpha(V)$ and $\lambda_1, \ldots, \lambda_m$ need not be distinct roots of P.

We first recall the definition of the class $D_\alpha(V)$ (see Definition 4 of [1]).

D e f i n i t i o n 1. Let L be a linear operator on V and $\alpha\in \ker L$ be fixed. If for all $u, v\in V$

$$L(uv)=uLv+vLu+\alpha LuLv$$

we say that $L\in D_\alpha(V)$.

Suppose that $L \in D_\alpha(V)$ and that $\lambda_1 = \cdots = \lambda_n = \lambda$ $(n \leqslant m)$ in Theorem 2. Then $v_1, \ldots, v_n$ in formula (2) should be replaced by $v, w_1 v, \ldots, w_{n-1} v$, where $v = v_1$ and $w_1, \ldots, w_{n-1}$ are recursively defined by

$$(3) \qquad w_0 = 1, \ Lw_k = \frac{k}{1+\alpha\lambda} w_{k-1} \qquad (k = 1, \ldots, n-1).$$

The proof is straightforward and is carried out in two steps. First, it can be proved by routine induction that for all $n \in N$

$$(4) \qquad L^p w_k v = \sum_{\nu=0}^{p} \binom{p}{\nu} k^{(\nu)} \lambda^{p-\nu} w_{k-\nu} v \qquad (p = 1, \ldots, n-1),$$

which implies that

$$P(L)\, w_k v = \sum_{\nu=0}^{k} \binom{k}{\nu} P^{(\nu)}(\lambda)\, w_{k-\nu} v.$$

However, since by hypothesis $P(\lambda) = \cdots = P^{(n-1)}(\lambda) = 0$, then clearly

$$P(L)\, w_k v = 0 \qquad (k = 0, 1, \ldots, n-1),$$

which shows that $v, w_1 v, \ldots, w_{n-1} v$ are solutions of (1).

Again by induction and by using (4) it can be shown that

$$\begin{vmatrix} v & w_1 v & \cdots & w_{n-1} v \\ Lv & L(w_1 v) & & L(w_{n-1} v) \\ \vdots & & & \\ L^{n-1} v & L^{n-1}(w_1 v) & & L^{n-1}(w_{n-1} v) \end{vmatrix} = (n-1)!\ (n-2)! \ldots 2!\ 1!\, v^n \neq 0,$$

i.e. that they are linearly independent.

An application of Theorem 1 completes the proof.

Examples. (i) If $L = \dfrac{d}{dt} : D \to D$ (D is the set all real differentiable functions), then $\alpha = 0$, and (3) becomes

$$\frac{d}{dt} w_k = k w_{k-1} \qquad (w_0 = 1;\ k = 1, \ldots, n-1),$$

which yields $w_k(t) = t^k$ $(k = 0, 1, \ldots, n-1)$.

(ii) If $L = \Delta : R^R \to R^R$ defined by $\Delta x(t) = x(t+1) - x(t)$, then $\alpha = 1$, and (3) becomes

$$\Delta w_k = \frac{k}{1+\lambda} w_{k-1} \qquad (w_0 = 1;\ k = 1, \ldots, n-1),$$

which yields

$$w_k(t) = (1+\lambda)^{-k} t^{(k)} \qquad (k = 0, 1, \ldots, n-1).$$

(iii) Let $L : R^R \to R^R$ be defined by $Lx(t) = x(at) - x(t)$ where $a > 0$ is fixed, and consider the functional equation

$$x(a^3 t) - 10\,x(a^2 t) + 33\,x(at) - 36\,x(t) = 0. \tag{5}$$

i.e.

$$P(L)\,x(t) = 0, \quad \text{where } P(\lambda) = \lambda^3 - 7\lambda^2 + 16\,\lambda - 12.$$

In this case $\alpha = 1$. The polynomial P has two roots: 3 and 2 (double root). The corresponding characteristic vectors are

$$t^{\log_a 4} \qquad \text{and} \qquad t^{\log_a 4},$$

which are linearly independent. In order to find the third linearly independent vector, we have to solve the equation

$$Lx(t) = 1/3,$$

which gives $x(t) = \frac{1}{3} \log_a t$, and hence the required vector is $\frac{1}{3}\, t^{\log_a 3} \log_a t$. Therefore, the general solution of the equation (5) is

$$x(t) = t^{\log_a 4}\,\varphi_1(\log_a t) + t^{\log_a 3}\,(\varphi_2(\log_a t) + \varphi_3(\log_a t)\log_a t)$$

where $\varphi_1, \varphi_2, \varphi_3$ are arbitrary functions, periodic with period 1.

REFERENCES

[1] J. D. Kečkić, *On some classes of linear equations*. Publ. Inst. Math. 24 (38) (1978), 89—97.

[2] J. D. Kečkić, *On some classes of linear equations*, II. Ibid. 26 (40) (1979), 135—144.