PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 31 (45) 1982, 73—81

REPRODUCTIVITY OF SOME EQUATIONS OF ANALYSIS

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1. Introduction

1.1. **Reproductive equations.** Following S.B. Prešić, we shall say that the equation in u:

$$u = Au \tag{1.1.1}$$

where $A : S \to S$ is a given mapping of a nonempty set into itself, is reproductive if

$$A(Au) = Au \text{ for every } u \in S, \quad \text{i.e.} \quad A^2 = A, \tag{1.1.2}$$

and the function A satisfying (1.1.2) will be called a reproductive function.

All the solutions (the general solution) of the reproductive equation (1.1.1) are given by the formula $u = At$, where $t \in S$ is arbitrary.

This suggests that a way to solve an equation in u:

$$u = Fu \tag{1.1.3}$$

where $F : S \to S$, would be to form a reproductive equation equivalent to (1.1.3).

Indeed, Prešić [1] showed that for any equation (1.1.3) which has at least one solution, it is possible to construct a reproductive equation which is equivalent to it. This general construction, however, is such that it cannot be used for solving the equation (1.1.3).

Nevertheless, in certain cases it is possible to construct a reproductive equation equivalent to a given equation (1.1.3) which will, as a result, yield an effective general solution of (1.1.3). Examples of such procedure may be found in Adamović [2], where a system of functional equations is written in the form of an equivalent reproductive equation, and in a number of papers by Prešić, but notably in [1], where he applied this idea to various kinds of equations (matrix equations, functional equations, Boolean equations, etc). Both mentioned authors obtained effective general solutions of the considered equations.

Reproductivity was not, as far as we know, applied to equations of analysis. In this note we shall apply reproductivity to ordinary differential equations, and in some subsequent notes we shall consider some other kinds of equations. Though we shall not, as a rule, obtain new results, the method used seems to be interesting, particularly as it will throw some light on the nature of the solutions of the equations in question.

Throughout this paper $D_k(I)$ will denote the set of all real functions which have the k-th order derivative on an interval I.

1.2. **Reproductivity and systems of equations.** In certain cases it is more convenient to form an equivalence between the given equation (1.1.3) and a disjunction of reproductive equations:

$$u=Fu \Leftrightarrow u=A_1u \vee \cdots \vee u=A_nu,$$

where $A_\nu^2=A_\nu$ ($\nu=1, \ldots, n$). In that case, the general solution of (1.1.3) can be expressed as

$$u=A_1t \vee \cdots \vee u=A_nt,$$

where $t \in S$ is arbitrary.

Sometimes a system of equations can be put into equivalence with one reproductive equation:

$$u=F_1u \wedge \cdots \wedge u=F_nu \Leftrightarrow u=Au \qquad (A^2=A)$$

In that case, the general solution of the system

$$u=F_1u \wedge \cdots \wedge u=F_nu$$

is given by $u=At$, where $t \in S$ is arbitrary.

1.3. **Constant equations.** A particularly important situation arises when A is a constant mapping, i.e. for all $u \in S$, $Au=u_0$, where $u_0 \in S$ is fixed. (The constant mapping is clearly reproductive). In that case the (general) solution of the corresponding equation (or system) is unique.

Having in mind the importance of unique solutions in various branches of analysis, we shall also be concerned with the following problem:

For a given equation (1.1.3), which is found to be equivalent to a reproductive equation (1.1.2), find what equations $(E_1), \ldots, (E_n)$ should be added to (1.1.3), so that the resulting system:

$$(1.1.3) \wedge (E_1) \wedge \cdots \wedge (E_n)$$

is equivalent to a constant equation.

It will be seen that the structure of the equivalent reproductive equation indicates the solution of this problem.

2. First order differential equations

2.1. **An auxiliary result.** Let $y \in D_1(I)$ and consider the equation

$$y'(x)=0. \tag{2.1.1}$$

According to the mean-value theorem, the equation (2.1.1) is equivalent to ·e equation

$$y(x)=y(x_0) \qquad (x_0 \in I \text{ is fixed}). \tag{1.2)}$$

The equation (2.1.2) is reproductive (for $A: D_1(I) \to D_1(I)$ defined by $Ay(x)=$ $=y(x_0)$, we clearly have $A^2=A$). Hence, its general solution is given by

$$y(x)=t(x_0) \qquad (t \in D_1(I) \text{ is arbitrary})$$

or, as we usually write

$$y=C \qquad (C \text{ arbitrary constant}).$$

The equivalent reproductive form (2.1.2) clearly shows that the system

(2.1.3) $$y'(x)=0 \wedge y(x_0)=y_0$$

(where $x_0 \in I$ and y_0 are given numbers) will be equivalent to the constant equation

(2.1.4) $$y(x)=y_0,$$

and hence the general (and unique) solution of (2.1.3) is (2.1.4).

2.2. **A class of first order equations**. In [3] we proved that the differential equation

(2.2.1) $$y'=f(x, y) \qquad (y \in D_1(I))$$

can be integrated by a given procedure if and only if it can be written as

(2.2.2) $$(g(x, y))'=0.$$

The equation (2.2.2) is equivalent to

(2.2.3) $$g(x, y(x))=g(x_0, y(x_0)) \qquad (x_0 \in I \text{ is fixed}).$$

Suppose that

$$g(u, v)=w \Leftrightarrow v=h(u, w).$$

Then (2.2.3) is equivalent to

(2.2.4) $$y(x)=h(x, g(x_0, y(x_0)))$$

and it is easily verified that the equation (2.2.4) is reproductive. Hence, its general solution is

$$y(x)=h(x, g(x_0, t(x_0))),$$

where $t \in D_1(I)$ is arbitrary, or $y=h(x, C)$, where C is an arbitrary constant.

The reproductive form (2.2.4) shows that uniqueness will be achieved if $y(x_0)$ is given. Indeed, the system

(2.2.5) $$y'=f(x, y) \wedge y(x_0)=y_0 \qquad (y_0 \text{ given real number})$$

is equivalent to the constant equation

(2.2.6) $$y(x)=h(x, g(x_0, y_0))$$

which means that (2.2.6) is the general (and the unique) solution of the system (2.2.5).

Cauchy's initial condition $y(x_0)=y_0$ is, therefore, a natural condition to add to (2.2.1) to ensure the uniqueness of solution,. However, as we shall see in the next section, it is not **the** natural condition for all first order equations, as we are sometimes led to believe by standard text-books.

2.3. **A special Clairaut's equation.** Let $y \in D_2(I)$ and consider the equation

(2.3.1) $$y=xy'+(y')^2.$$

If $x_0 \in I$ is fixed (but arbitrary), from (2.3.1) follows

(2.3.2) $$y(x_0)=x_0 y'(x_0)+y'(x_0)^2.$$

Differentiating (2.3.1) we obtain

$$y''(x+2y')=0,$$

and we distinguish between two cases:

(i) $y''=0$. Then $y^{(3)}=y^{(4)}=\cdots=0$.

(ii) $x+2y'=0$. Then

$$y'(x_0)=-x_0/2, \tag{2.3.3}$$

and $y''=-1/2$, $y^{(3)}=y^{(4)}=\cdots=0$.

If we develop y into Taylor's series in a neighbourhood of x_0, we get:

(i) $y(x)=y(x_0)+(x-x_0)\,y'(x_0)$,

i.e. having in mind (2.3.2):

$$y(x)=y'(x_0)^2+xy'(x_0). \tag{2.3.4}$$

(ii) $y(x)=y(x_0)+(x-x_0)\,y'(x_0)+\dfrac{1}{2}(x-x_0)^2\,y''(x_0)$,

or, using (2.3.2) and (2.3.3):

$$y(x)=\left(-\frac{1}{2}x_0+\frac{1}{4}x_0^2\right)-\frac{1}{2}(x-x_0)\,x_0-\frac{1}{4}(x-x_0)^2,$$

i.e.

$$y(x)=-x^2/4. \tag{2.3.5}$$

The equations (2.3.4) and (2.3.5) are reproductive equations (the second being also a constant equation). Since for $y\in D_2(I)$ we have

$$(2.3.1)\Leftrightarrow(2.3.4)\vee(2.3.5)$$

we conclude that the general solution of (2.3.1) is given by

$$y(x)=t'(x_0)^2+xt'(x_0)\vee y(x)=-x^2/4,$$

where $t\in D_2(I)$ is arbitrary, or by

$$y(x)=C^2+Cx\vee y(x)=-x^2/4,$$

where C is an arbitrary constant.

Remark. It is customary to say that the general solution of (2.3.1) is $y=Cx+C^2$ (C arbitrary constant) and that its singular solution is $y=-x^2/4$.

Throughout this paper we have taken the general solution to mean the solution containing all the solutions, which is a natural definition, though not in accordance with tradition.

The equivalent reproductive form of (2.3.1), provided that $y\in D_2(I)$, namely

$$y(x)=y'(x_0)^2+xy'(x_0)\vee y(x)=-x^2/4,$$

shows what conditions should be added to the equation (2.3.1) to ensure the uniqueness of solution.

The additional condition is clearly

$$y'(x_0)=y_0' \qquad \left(y_0'\left(\neq -\frac{1}{2}x_0\right) \text{ given number}\right).$$

This shows that Cauchy's initial condition $y(x_0)=y_0$ is not a natural condition for Clairaut's equation (2.3.1).

Remark. If $y_0'=-x_0/2$, then there will be two solutions satisfying (2.3.1) and the given condition $y'(x_0)=y_0'$, and hence some more conditions should be added, if we want to secure a unique solution.

3. Linear second order differential equations

3.1. **The general solution.** If $f, g\in D_1(I)$, let

$$W(f,g)=W(f,g;x)=f'(x)g(x)-f(x)g'(x).$$

It is easily verified that for three functions $f, g, h\in D_1(I)$, the following identity is valid:

$$W(f,g)h+W(h,f)g+W(g,h)f\equiv 0 \qquad (x\in I). \tag{3.1.1}$$

Let $y\in D_2(I)$ and consider the equation

$$y''+p(x)y'+q(x)y=0 \qquad (x\in I). \tag{3.1.2}$$

Suppose that f and g are linearly independent solutions of (3.1.2), i.e. $W(f,g;x)\neq 0$ $(x\in I)$. If y is any other solution of (3.1.2), from (3.1.1) we get

$$y(x)=\frac{W(y,g;x)}{W(f,g;x)}f(x)+\frac{W(f,y;x)}{W(f,g;x)}g(x). \tag{3.1.3}$$

Moreover, if u and v are solutions of (3.1.2) then

$$\frac{d}{dx}(W(u,v;x))+p(x)W(u,v;x)=0,$$

which implies that the quotients appearing on the right hand side of (3.1.3) do not depend on x, provided that p has a primitive function on I. Hence, (3.1.3) can be written in the form

$$y(x)=\frac{W(y,g;a)}{W(f,g;a)}f(x)+\frac{W(f,y;b)}{W(f,g;b)}g(x), \tag{3.1.4}$$

where a, $b\in I$ are fixed.

The equations (3.1.2) and (3.1.4) are equivalent. However, the equation (3.1.4) is reproductive, since it is easily seen that for the mapping $A: D_2(I)\to D_2(I)$ defined by

$$Ay(x)=\frac{W(y,g;a)}{W(f,g;a)}f(x)+\frac{W(f,y;b)}{W(f,g;b)}g(x),$$

we have $A^2=A$. This means that the general solution of the equation (3.1.4), and hence of (3.1.2), is

$$y(x)=\frac{W(t, g; a)}{W(f, g; a)}f(x)+\frac{W(f, t; b)}{W(f, g; b)}g(x), \tag{3.1.5}$$

where $t\in D_2(I)$ is arbitrary, or

$$y(x)=C_1 f(x)+C_2 g(x), \tag{3.1.6}$$

where C_1 and C_2 are arbitrary constants.

Remark. In the last step of the proof, i.e. in the transitition from (3.1.5) to (3.1.6) it is necessary to show that the quotients $\frac{W(t, g; a)}{W(f, g; a)}$ and $\frac{W(f, t; b)}{W(f, g; b)}$ can take arbitrary real values. However, we have proved that (3.1.2) $\Rightarrow$ (3.1.6) where C_1 and C_2 are constants, and the implication (3.1.6) $\Rightarrow$ (3.1.2) where C_1 and C_2 are arbitrary constants is directly verified.

In other words we have proved the following theorem:

If f and g are linearly independent solutions of the equation (3.1.2), where p has a primitive function on I, then the general solution of that equation is given by (3.1.6).

The above theorem is, in a way, often taken for granted. In fact, we could not find a proof of this theorem, *as it stands*, anywhere.

The standard text-book procedure runs roughly as follows:

(i) It is first necessary to prove an existence theorem such as: If p and q are continuous on I, there exists a unique solution Y of (3.1.2) which satisfies the conditions $Y(x_0)=Y_0$, $Y'(x_0)=Y_0'$, where $x_0\in I$, Y_0 and Y_0' are given numbers.

(ii) Suppose that Y is a solution of (3.1.2). According to (i), Y is the only solution of (3.1.2) satisfying $Y(x_0)=Y_0$ and $Y'(x_0)=Y_0'$, where $x_0\in I$ is fixed. But the solution (3.1.6) with

$$C_1=\frac{Y_0' g(x_0)-Y_0 g'(x)}{W(f, g; x_0)}, \qquad C_2=\frac{Y_0 f'(x_0)-Y_0' f(x_0)}{W(f, g; x_0)}$$

also satisfies (3.1.2) and $Y(x_0)=Y_0$, $Y'(x_0)=Y_0'$, and hence it must coincide with Y.

The proof of the above theorem by means of reproductivity has several advantages over the standard method sketched above:

(i) The proof does not require the theorem on the existence of the unique Cauchy solution of (3.1.2);

(ii) The restrictions needed for the coefficients p and q are weaker: we need only suppose that p has a primitive function on I;

(iii) Cauchy's initial conditions are not used in the proof. Indeed, as we shall see later, Cauchy's conditions naturally follow from the reproductive form (3.1.4).

The reproductive proof of the above theorem is easily extended to n-th order linear equations.

The equation (3.1.4) is not the only reproductive equation equivalent to (3.1.2). For example, from (3.1.4), interchanging a and b, and adding the obtained equations, we get

$$(3.1.7)\qquad y(x)=\frac{W(f,g;b)\,W(y,g;a)+W(f,g;a)\,W(y,g;b)}{2\,W(f,g;a)\,W(f,g;b)}f(x)+$$
$$+\frac{W(f,g;a)\,W(f,y;b)+W(f,g;b)\,W(f,y;a)}{2\,W(f,g;a)\,W(f,g;b)}g(x).$$

It is easily shown that the equation (3.1.7), equivalent to (3.1.2), is reproductive.

3.2. **Cauchy's solution**. Putting $a=b=x_0$ into (3.1.4) we get

$$(3.2.1)\qquad y(x)=\frac{y'(x_0)\,g(x_0)-y(x_0)\,g'(x_0)}{f'(x_0)\,g(x_0)-f(x_0)\,g'(x_0)}f(x)+\frac{f'(x_0)\,y(x_0)-f(x_0)\,y'(x_0)}{f'(x_0)\,g(x_0)-f(x_0)\,g'(x_0)}g(x).$$

The equivalent reproductive equation (3.2.1) indicates what additional equations should be added to (3.1.2) to ensure the uniqueness of solution, i.e. to transform (3.2.1) into a constant equation.

Clearly, $y(x_0)$ and $y'(x_0)$ should be given.

In fact, the system

$$(3.2.2)\qquad (3.1.2)\wedge y(x_0)=y_0\wedge y'(x_0)=y_0'$$

where $x_0\in I$, y_0, y'_0 are given numbers, is equivalent to the constant equation

$$(3.2.3)\qquad y(x)=\frac{y_0'\,g(x_0)-y_0\,g'(x_0)}{f'(x_0)\,g(x_0)-f(x_0)\,g'(x_0)}f(x)+\frac{y_0 f'(x_0)-y_0' f(x_0)}{f'(x_0)\,g(x_0)-f(x_0)\,g'(x_0)}g(x),$$

and hence (3.2.3) is the general (and the unique) solution of the system (3.2.2).

Remark. This procedure emphasizes the natural need for Cauchy's initial conditions $y(x_0)=y_0$, $y'(x_0)=y_0'$, since they ensure the uniqueness of solution for any equation (3.1.2).

Remark. The solution (3.2.3) is obtained from the general solution (3.1.5) by putting $a=b=x_0$ and $t(x)=y_0+y_0'(x-x_0)$, though this is not the only choice of t.

3.3. **Examples of boundary value problems.** By considering the equation

$$(3.3.1)\qquad y''+y=0$$

we shall illustrate how the reproductive forms (3.1.4) and (3.1.7) can be used to yield general solutions of some boundary value problems, which will in certain cases be unique.

Clearly, for the equation (3.3.1) we may take

$$f(x)=\sin x,\qquad g(x)=\cos x,$$

so that $W(f, g)=1$. The reproductive equations (3.1.4) and (3.1.7) become

(3.3.2) $$y(x)=[y'(a)\cos a+y(a)\sin a]\sin x+[y(b)\cos b-y'(b)\sin b]\cos x$$

and

(3.3.3) $$\begin{aligned}2y(x)=&(y'(a)\cos a+y(a)\sin a+y'(b)\cos b+y(b)\sin b)\sin x\\&+(y(b)\cos b-y'(b)\sin b+y(a)\cos a-y'(a)\sin a)\cos x,\end{aligned}$$

respectively.

(i) Consider the problem

(3.3.4) $$(3.3.1)\wedge y(0)=0\wedge y(\pi)=0.$$

Putting $a=0$, $b=\pi$ into (3.3.2), we get

$$y(x)=y'(0)\sin x-y(\pi)\cos x,$$

i.e. using $y(\pi)=0$.

(3.3.5) $$y(x)=y'(0)\sin x,$$

and it is easily seen that (3.3.4) $\Leftrightarrow$ (3.3.5).

Since (3.3.5) is reproductive, the general solution of the system (3.3.4) is given by

$$y(x)=t'(0)\sin x \qquad (t\in D_2(I) \text{ is arbitrary})$$

or $y(x)=C\sin x$ (C arbitrary constant).

(ii) For the problem

(3.3.6) $$(3.3.1)\wedge y(0)=0\wedge y(\pi)=1,$$

using the same procedure, we arrive at the reproductive equation

$$y(x)=y'(0)\sin x-\cos x,$$

but this equation is not equivalent to (3.3.6), indicating that (3.3.6) has no solutions.

(iii) The problem

(3.3.7) $$(3.3.1)\wedge y(\pi)=\lambda\wedge y'(0)=\nu$$

(λ, ν given numbers) has a unique solution. Indeed, the equation (3.3.2) for $a=0$, $b=\pi$ becomes

$$y(x)=y'(0)\sin x-y(\pi)\cos x,$$

and using $y(\pi)=\lambda$, $y'(0)=\nu$,

(3.3.8) $$y(x)=\nu\sin x-\lambda\cos x.$$

Since the constant equation (3.3.8) is equivalent to the system (3.3.7), the unique solution of (3.3.7) is (3.3.8).

(iv) Finally, consider the problem

(3.3.9) $$(3.3.1)\wedge y(0)-y(\pi)=\lambda\wedge y'(0)-y'(\pi)=\nu$$

(λ, ν given numbers). The reproductive equation (3.3.3) for $a=0$, $b=\pi$ becomes

$$2\,y(x)=(y'(0)-y'(\pi))\sin x+(y(0)-y(\pi))\cos x,$$

and using (3.3.9)

$$y(x)=(\nu\sin x+\lambda\cos x)/2. \tag{3.3.10}$$

The constant equation (3.3.10) is equivalent to (3.3.9), and hence (3.3.10) is the unique solution of (3.3.9).

The author is indebted to Professor D.D. Adamović for careful reading of the first version of this paper, and for a number of valuable comments and suggestions.

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