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MEASURABLE OUTER KERNELS OF SETS

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Abstract. The notion of a measurable outer kernel K of a set S is introduced as a measurable superset K of S such that the only measurable subsets of $K-S$ are of zero measure. Some theorems concerning the union of measurable outer kernels of sets and the masurable outer kernels of the union of these sets are proved.

In what follows every set is a subset of the set R of all real numbers and every notion referring to outer measure and measure is in the sense of Lebesgue. Moreover, as usual $m^*(E)$ and $m(E)$ denote respectively the outer measure and the measure (if it exists) of E.

Let S be a set and K a measurable set such that $S \subseteq K$ and such that the only measurable subsets of $K-S$ are of zero measure. If such a set K exists we call it a *measurable outer kernel of* S. Thus, if K exists it is, in a sense, a minimal measurable set which contains S. Clearly, if K is a measurable outer kernel of S then for every set Z of measure zero $K \cup Z$ is also a measurable outer kernel of S. Hence if S has a measurable outer kernel, the latter is not unique. Obviously, if K_1 and K_2 are measurable outer kernels of S then $K_1 - K_2$ as well as $K_2 - K_1$ is of zero measure.

Theorem 1. *Let S be a set then a measurable outer kernel of S exists.*

Proof. Let M be the set of all the supersets of S that are measurable, i. e.,

$$M = \{X \mid S \subseteq M \text{ and } X \text{ is measurable}\}, \tag{1}$$

First, we consider the case where:

$$m = \inf_{X \in M} m(X) < \infty. \tag{2}$$

Clearly, we may select (in various ways) a denumerable number of elements $Y_0, Y_1, Y_2, \ldots$ of M such that

$$m = \inf_{i \in \omega} m(Y_i) \quad \text{where } Y_i \in M \text{ for every } i \in \omega. \tag{3}$$

We claim that the set K given by:

$$K = \bigcap_{i \in \omega} Y_i \tag{4}$$

is a measurable outer kernel of S.

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Indeed, as (4) shows, K is a denumerable intersection of measurable sets Y_i and therefore K is a measurable set. Also, from (1) and (4) it follows that $S \subseteq K$. Thus, K is a measurable superset of S. As such $K \in M$ and by (2) we have $m \leqslant m(K)$. On the other hand, by (4) we have $m(K) \leqslant m(Y_i)$ for every $i \in \omega$ which by (3) implies $m(K) \leqslant m$. Hence,

(5) $$m(K) = m = \inf_{X \in M} m(X).$$

Since K is a measurable superset of S, to complete the proof for the case (2), it remains to show that every measurable subset Z of $K - S$ is of zero measure. Clearly, $(K - Z) \in M$ and therefore by (2) we have $m \leqslant m(K - Z) = m(K) - m(Z)$ which by (5) implies $m(Z) = 0$, as desired.

Now, we consider the case where:

$$m = \inf_{X \in M} m(X) = \infty.$$

For every $n = 0, \pm 1, \pm 2, \ldots$, let S_n denote the intersection of S with the closed unit real interval $[n, n+1]$. From (2) it follows that every S_n has a measurable outer kernel K_n. But then it can be readily verified that $\cup \{K_n \mid n = 0, \pm 1, \pm 2, \ldots\}$ is a measurable outer kernel of S, as desired.

Next, we prove:

Theorem 2. *Let K be a measurable outer kernel of S. Then*

(6) $$m^*(S) = m(K)$$

Proof. Since $S \subseteq K$ we see that $m^*(S) \leqslant m(K)$. On the other hand, $m^*(S)$ is the infimum of the measures of all the open sets which cover S and therefore by (1) and (2) we have $m \leqslant m^*(S)$. Thus, $m^*(S) = m$. But then (6) follows from (5).

Combining (5) and (6) we have:

(7) $$m^*(S) = m(K) = \inf_{X \in M} m(X)$$

Next we prove a converse of Theorem 2.

Theorem 3. *Let H be a measurable superset of S such that $m^*(S) = m(H)$. Then H is a measurable outer kernel of S.*

Proof. It is enough to show that every measurable subset Z of $H - S$ is of zero measure. Since $S \subseteq (H - Z)$ we see that $m^*(S) \leqslant m(H - Z) = m(H) - m(Z)$. But then the hypothesis $m^*(S) = m(H)$ implies $m(Z) = 0$, as desired.

Combining Theorems 2 and 3, we have:

Theorem 4. *A measurable superset K of a set S is a measurable outer kernel of S if and only if $m^*(S) = m(K)$.*

Next we prove:

Theorem 5. *The union of measurable outer kernels of a denumerable number of sets is a measurable outer kernel of the union of these sets.*

P r o o f. For every $i \in \omega$, let K_i be a measurable outer kernel of S_i. We show that $\bigcup_{i\in\omega} K_i$ is a measurable outer kernel of $\bigcup_{i\in\omega} S_i$. To this end (since $\bigcup_{i\in\omega} K_i$ is obviously a measurable superset of $\bigcup_{i\in\omega} S_i$) it is enough to prove that if Z is a measurable set and $Z \subseteq (\bigcup_{i\in\omega} K_i - \bigcup_{i\in\omega} S_i)$ then $m(Z) = 0$. But in this case $Z \subseteq \bigcup_{i\in\omega}(K_i - \bigcup_{i\in\omega} S_i)$ and therefore

$$Z = \bigcup_{i\in\omega}(Z \cap (K_i - \bigcup_{i\in\omega} S_i)). \tag{8}$$

However, for every $i \in \omega$ we have $Z \cap (K_i - \bigcup_{i\in\omega} S_i) = Z \cap K_i \subseteq (K_i - S_i)$. Thus, for every $i \in \omega$ we see that $Z \cap K_i$ is a measurable subset of $K_i - S_i$ and consequenly, $m(Z \cap K_i) = 0$. But then (8) shows that Z is a denumerable union of sets of zero mesure and therefore $m(Z) = 0$, as desired.

R e m a r k 1. The statement of Theorem 5 does not remain valid if in it „denumerable" is replaced by „nondenumerable." In fact, examples can be given where the union of measurable outer kernels of sets not only is not a measurable outer kernel of the union of these sets but is not even measurable. For instance, let $N = \{a, b, c, \ldots\}$ be a nonmeasurable set. Clearly, every singleton $\{a\}, \{b\}, \{c\}, \ldots$ is a measurable outer kernel of itself. However the union of these singletons is N which is nonmeasurable and hence cannot be a measurable outer kernel of any set.

Also, examples can be given to show that the statement of Theorem 5 does not remain valid if in it „denumerable" is replaced by the first nondenumerable cardinal $\aleph_1$ (even with the provision that $\aleph_1 < 2^{\aleph_0}$). This is because there are models for ZF in which $\aleph_1 < 2^{\aleph_0}$ and in which there are nonmeasurable sets of cardinality $\aleph_1$ (namely, the set of constructible reals).

However, in spite of Theorem 5 we have the following:

T h e o r e m 6. *Let $\aleph$ be any cardinal such that $\aleph < 2^{\aleph_0}$. Then it is consistent (with the usual axioms of* ZF) *to assume that the union of measurable outer kernels of $\aleph$ many sets is a measurable outer kernel of the union of these sets.*

P r o o f. It is known [1, p. 114] that Martin's axiom is consistent with the usual axioms of ZF. Also it is known [2, p. 167] that Martin's axiom implies that if $\aleph < 2^{\aleph_0}$ then (i) the union of $\aleph$ many measurable sets is measurable and (ii) the union of $\aleph$ many sets of zero measure is of zero measure. But then based on (i) and (ii), a proof of the Theorem can be given by the proof of Theorem 5 where the denumerable cardinal ω is replaced everywhere by $\aleph$.

As an application of the notion of the measurable outer kernel, we prove the following theorem which reduces the question of the measurability of the intersection (as well as the union) of $\aleph$ many (where $\aleph$ is any cardinal) measurable sets to the question of zero-measuredness of the union of $\aleph$ many sets of zero measure.

Theorem 7. *Let $\aleph$ be a cardinal. Let us assume that the union of any $\aleph$ many sets of zero measure is of zero measure. Then the intersection (as well as the union) of any $\aleph$ many measurable sets is measurable.*

Proof. Let $(M_i)_{i\in\aleph}$ be a set of $\aleph$ many measurable sets M_i. Let K be a measurable outher kernel of $\bigcap_{i\in\aleph} M_i$. Clearly,

$$K-\bigcup_{i\in\aleph}(K-M_i)=\bigcap_{i\in\aleph} M_i. \tag{9}$$

However, for every $i\in\aleph$ wə see that $K-M_i$ is a measurable subset of $K-\bigcap_{i\in\aleph} M_i$ and thus $K-M_i$ is of zero measure. But then, by the assumption of the Theorem, $\bigcup_{i\in\aleph}(K-M_i)$ is of zero measure, which by (9) implies that $\bigcap_{i\in\aleph} M_i$ is measurable, as desired. The measurability of $\bigcup_{i\in\aleph} M_i$ follows from the fact that by virtue of the Theorem $\bigcap_{i\in\aleph}(R-M_i)$ is measurable which implies that $R-\bigcap_{i\in\aleph}(R-M_i)=\bigcup_{i\in\aleph} M_i$ is also measurable.

Remark 2. As expected, „a measurable inner kernel Q of a set S" can be introduced as the dual of „a measurable outer kernel of S" by defining Q to be a measurable subset of S such that the only measurable subsets of $S-Q$ are of zero measure Also, as expected the duals of Theorems 1 to 6 then can be stated and proved. The dual of Theorem 7 is stated in the statement of the Theorem parenthetically.

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