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ON SOME INEQUALITIES FOR QUASI-MONOTONE SEQUENCES

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0. Let $p \neq 0$ be a real constant. The operator L_p will be defined in the following way (see [**1**]):

$$L_p(a_n) = a_{n+1} - pa_n \quad (n \in N).$$

For a sequence $a = (a_n)$ we shall say that it is p-monotone or that it belongs to the class K_p if the inequality

$$L_p(a_n) \geq 0$$

is valid for all $n \in N$.

The following theorem is given in [**1**];

THEOREM A. *Let for a given sequence $a = (a_n)$ the sequaence $A = (A_n)$ be defined by*

$$A_n = \frac{p_1 a_1 + \cdots + p_n a_n}{p_1 + \cdots + p_n}.$$

(i) *If we have $p = q$ then the implication*

$$a \in K_n \Rightarrow A \in K_q \tag{1}$$

holds true for every sequence of the class K_p and for arbitrary positive weights $p = (p_n)$ if and only if $p = q = 1$.

(ii) *If p and q satisfy one of the conditions*

$$p > q > 1; \ 0 < p < q < 1; \quad or \quad p < q < 0,$$

then implication (1) *holds true for an arbitrary sequence of the class K_p if and only if the sequence $p = (p_n)$ of positive weights is given by*

$$p_n = p_1 \frac{qn - 1 - q^{n-2}}{p^{n-1} - q^{n-2}} \prod_{k=1}^{n-1} \frac{p^k - q^{k-1}}{p^k - q^k} \quad (n = 2, 3, \dots)$$

where the weight p_1 is an arbitrary given positive number.

1. In this paper we shall shown that some inequalities for monotone sequences are also valid for p-monotone sequences, i.e. we shall give the necessary and sufficien conditions for the validity of these inequalities.

First, we shall notice that the following identity follows, from the well-known Abel identity:

$$\sum_{i=1}^{n} w_i a_i = a_1 \sum_{i=1}^{n} p^{i-1} w_i \sum_{k=2}^{n} \Big(\sum_{i=k}^{n} p^{i-k} w_i \Big) L_p(a_{k-1}). \tag{2}$$

Using (2), we can easily obtain the following theorem:

THEOREM 1. *Let $w = (w_n)$ be an arbitrary real sequence.*

(a) *Inequality*

$$\sum_{i=1}^{n} w_i a_i \geq 0 \tag{3}$$

holds for every sequence a from K_p if and only if

$$\sum_{i=1}^{n} p^{i-1} w_i = 0$$

and

$$\sum_{i=k}^{n} p^{i-k} w_i \geq 0 \quad (k = 2, \dots, n).$$

(b) *Inequality* (3) *holds for every sequence a from K_p such that $a_1 \geq 0$ if and only if*

$$\sum_{i=k}^{n} p^{i-k} w_i \geq 0 \quad (k = 1, \dots, n).$$

2. Let $a \in K_p$, $b \in K_q$ be real sequences, and let $x_{ij} (i = 1, \dots, n; j = 1, \dots, m)$ be real numbers. Then necessary and suficient conditions for the numbers x_{ij}, such that the inequality

$$F(a, b) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} a_i b_j \geq 0 \tag{4}$$

holds: 1° for every p-monotone sequence a and for every q-monotone sequence b, or 2° for every p-monotone sequence a and q-monotone sequence b such that $a_1 \geq 0$ and $b_1 \geq 0$, are contained in the following theorem:

THEOREM 2. (a) *With the condition 1°, $F(a, b) \geq 0$ if and only if*

$$\begin{gathered} X_{1,s} = 0,\ (s = 1, \dots, m), \quad X_{r,1} = 0\ (r = 2, \dots, n) \\ X_{r,s} \geq 0\ (r = 2, \dots, n;\ s = 2, \dots, m), \end{gathered} \tag{5}$$

where

$$X_{r,s}=\sum_{i=r}^{n}\sum_{j=s}^{m}p^{i-r}q^{j-s}x_{ij}.$$

(b) *With the condition* 2°, $F(a,b)\geq 0$, *if and only if*

$$X_{r,s}\geq 0\ \ (r=1,\dots,n;\ \ s=1,\dots,m).$$

Proof. (a) (i) Let $a_i=0$ $(i=1,\dots,r-1)$ $a_i=p^{i-r}(i=r,\dots,n)$, and let $b_j=q^{j-1}(j=1,\dots,m)$ or $b_j=-q^{j-1}(j=1,\dots,m)$. Then from (4), we get the condition $X_{r,1}=0$. By analogy, we get $X_{1,s}=0$. Now, let

$$\begin{aligned}a_i=&0\,(i=1,\dots,\ r-1)\quad a_i=p^{i-r}(i=r,\dots,n),\\ b_j=&0\,(j=1,\dots,s-1)\quad b_j=p^{j-s}(j=s,\dots,m).\end{aligned}$$

Then, from (4), we get $X_{r,s}\geq 0$. So condition (5) is necessary.

(ii) Let be $s_j=\sum\limits_{i=1}^{n}x_{ij}a_i$. Then

$$f(a,b)=\sum_{j=1}^{m}s_jb_j=b_1\sum_{j=1}^{m}q^{j-1}s_j+\sum_{s=2}^{m}\left(\sum_{j=s}^{m}q^{j-s}s_j\right)L_q(b_{s-1}).$$

Now, we write $x_i=\sum\limits_{j=s}^{m}q^{j-s}x_{ij}$. Then

$$\begin{aligned}\sum_{j=s}^{m}q^{j-s}s_j=&\sum_{i=1}^{n}\left(\sum_{j=s}^{m}q^{j-s}x_{ij}\right)a_i=\sum_{i=1}^{n}x_ia_i\\ =&a_1\sum_{i=1}^{n}p^{i-1}x_i+\sum_{r=2}^{n}\left(\sum_{i=r}^{n}p^{i-r}x_i\right)L_p(a_{r-1})\\ =&a_1X_{1,s}+\sum_{r=2}^{n}X_{r,s}L_p(a_{r-1}).\end{aligned}$$

For $s=1$, we have

$$\sum_{j=1}^{m}q^{j-1}s_j=a_1X_{1,1}+\sum_{r=2}^{n}X_{r,1}L_p(a_{r-1}).$$

So,

$$\begin{aligned}F(a,b)=a_1b_1X_{1,1}&+b_1\sum_{r=2}^{n}X_{r,1}L_p(a_{r-1})+a_1\sum_{s=2}^{m}X_{1,s}L_q(b_{s-1})\\ &+\sum_{r=2}^{n}\sum_{s=2}^{m}X_{r,s}L_p(a_{r-1})L_q(b_{s-1}).\end{aligned}$$

Based on (5), it is evident that (4) holds.

Analogously we can prove (b).

Remark. *For $p = q = 1$, we have the result from* [**2**].

Analogously to the previous proof (see also [**3**]), we can prove the following theorem:

Theorem 3. *Let $a_j = (a_{j1}, \dots, a_{jn})$ $(j = 1, \dots, m)$ be real sequences and let $x_{j_1} \dots j_m (j_k = 1, \dots, n_k,\ k = 1, \dots, m)$ be real numbers. Then necessary and sufficient conditions for the numbers $x_{j_1} \dots j_m$, for the inequality*

$$\sum_{j_1=1}^{n_1} \cdots \sum_{j_1=1}^{n_m} x_{j_1} \dots j_m a_{1j_1} \cdots a_{mj_m} \geq 0$$

to hold for every p_j-monotone sequence a_j such that $a_{j1} \geq 0$ $(j = 1, \dots, m)$ are

$$\sum_{j_1=s_1}^{n_1} \cdots \sum_{j_m=s_m}^{n_m} p_1^{j_1-s_1} \cdots p_m^{j_m-s_m} x_{j_1} \dots j_m \geq 0$$

for $j_k = 1, \dots, n_k,\ k = 1, \dots, m$.

Remark. *For $p_1 = \cdots = p_m = 1$, we have the result from* [**3**].

3. Now, we shall give a generalization of Theorem A.

Let us consider a triangular matrix of real numbers $(p_{n,i})$ (where $=$ 0, 1, $\dots, n$; $n = 0, 1, \dots$). Let us define the sequence (σ_n), for a given sequence (a_n) by

$$\sigma_n = \sum_{j=0}^{n} p_{n,n-j} a_j. \tag{5}$$

Then the following theorem holds:

Theorem 4. *A necessary and sufficient condition for the implication*

$$(a_n) \in K_p \quad \Rightarrow \quad (\sigma_n) \in K_q$$

to be valid, for every sequence (a_n), where the sequence (σ_n) is given by (5), is that the following conditions, for every n,

$$d_{n,n} - q d_{n-1,n-1} = 0,\ d_{n,n-k} - q d_{n-1,n-k-1} \geq 0\ (k = 1, \dots, n-1)$$
$$d_{n,0} \geq 0,$$

hold, where

$$d_{n,k} = \sum_{j=0}^{k} p^{k-j} p_n, j.$$

Proof. We have

$$L_q(\sigma_{n-1}) = \sigma_n - q\sigma_{n-1} = \sum_{j=0}^{n} p_{n,n-j}a_j - q\sum_{j=0}^{n-1} p_{n-1,n-1-j}a_j = \sum_{j=0}^{n} w_j a_j$$

where $w_j = p_{n,n-j} - qp_{n-1,n-1-j} (j = 0, 1, \ldots, n-1)$ and $w_n = p_{n,0}$. Using Theorem 1, we obtain Theorem 4.

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