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INVARIANCE OF SPECTRAL TYPE OF A STOCHASTIC PROCESS WITH RESPECT TO TRANSFORMATION OF TIME

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The problem of isometric for families of subspaces respectively the problem of invariance of spectral type of a stochastic process with respect to linear transformation is discussed in [**5**] and [**8**]. For a process $\{x(t);\, t \in I\}$ which has a discrete spectral type it was proved in [**8**] that the families $\{\mathcal{H}_t(x);\, t \in I\}$ and $\{\mathcal{H}_t(y) = A\mathcal{H}_t(x), t \in I\}$ are isometric if $A : \mathcal{H}(x) \to \mathcal{H}(y)$ is a bounded linear transformation with a bounded inverse $A^{-1} : \mathcal{H}(y) \to \mathcal{H}(x)$.

In this work we shall solve the problem of spectral type invariance of a stohastic process with respect to a transformation of time, that is we shall give the answer to the question:

Which properties of a continuous nondecreasing function φ, which maps the interval $I_1 = [t_1, \infty\rangle$ onto $I = [t_0, \infty\rangle$, assure that the stochactics processes $\{x(t);\ t \in I\}$ and $\{y(t) = x(\varphi(t));\ t \in I_1\}$ have the same spectral type?

In the first part of the article we shall state some known results from the theory of spectral multiplicity of stochastic processes, which will be used in further work. In the second part the formulated problem is solved under the assumption that φ maps I onto I, finally the third part deals with general problem.

I. Stochastic process of the second order $\{x(t);\ t \in I\}$, $I = [t_0, \infty\rangle$, $(Ex(t) = 0$ and $E|x(t)|^2 < \infty)$ can be considered as a curve in the Hilbert space $\mathcal{H}$ of random variables y with $Ey = 0,\ E|y|^2 < \infty$. The inner product in $\mathcal{H}$ is $(y_1, y_2) = Ey_1 \cdot \tilde{y}_2$.

The process $\{x(t);\ t \in I\}$ is represented by the subspace

$$\mathcal{H}_t(x) = \overline{\mathcal{L}\{x(s);\ s \le t\}},$$
$$\mathcal{H}(x) = \mathcal{H}_\infty(x),\ \mathcal{H}_{t_0}(x) = \bigcap_{t \in I} \mathcal{H}_t(x)$$

Suppose that

(A) $\mathcal{H}(x)$ is a separable space and

(B) $\mathcal{H}_{t_0}(x) = 0$ (that is: the process $\{x(t);\ t \in I\}$ is purely nondeterministic).

Then the stochastic proces $\{x(t), t \in I\}$ has the Hida-Cramér representation ([**5**]):

$$x(t) = \sum_{n=1}^{N} \int_{t_0}^{t} g_n(t,u)dz_n(u), \quad t \in I, \tag{1}$$

where:

— $\sum_{n=1}^{N} \int_{t_0}^{t} |g_n(t,u)|^2 \, dF^x_{z_n}(u) < \infty \ \ t \in I$;

— $N \leq \infty$ is the multiplicity of the process;

— $(\{z_n(t);\ t \in I\})_{n=\overline{1,N}}$ is a sequence of mutually orthogonal stochastic processes with orthogonal increments;

— $t \to F^x_{z_n}(t) = \|z_n(t)\|^2, t \in I$, are the distribution functions which induce the measures ordered in the sense of absolute continuity belonging to the spectral type of process.

$$dF^x : dF^x_{z_1} \geq dF^x_{z_2} \geq \cdots \geq dF^x_{z_n}; \tag{2}$$

— the family of functions $\{(g_n(t,u);\ u \leq t),\ n = \overline{1,N}$, the parameter $t \in I\}$ is complete in the space $\mathcal{L}_2(dF_{z_1}; I)$ in sense of definition 2 [**5**].

The most important result for application of spectral multiplicity theory in Hilbert spaces in the theory of stochastic processes is given in [**1**] and [**5**]. It says:

For given sequence of spectral types:

$$dF : dF_1 > dF_2 > \cdots > dF_N, \quad N \leq \infty,$$

there exists a stochastic process $\{x(t); t \in I\}$ of second order, which satisfies the conditions (A) and (B) and which has spectral type dF.

In [**1**] it is shown that there exists even an harmonic process $\{x(t);\ t \in I\}$ for which $dF^x = dF$. It is proved (also in [**1**]) that the spectral type of a stohastic process $\{x(t); t \in I\}$ is uniquelly determined by the correlation function, defined by:

$$r_x(u,v) = Ex(u) \cdot \overline{x(v)} = (x(u),\, x(v)) \quad u, v \in I. \tag{3}$$

The converse of that theorem in general case is not true, because two stochastic processes with equal spectral type can have different correlation functions. The spectral type

$$dF^y : dF^y_{z_1} > dF^y_{z_2} > \cdots > dF^y_{z_M}, \quad M \leq \infty,$$

of the process $\{y(t); t \in I\}$ is equal to the spectral type (2) of a stochastic process $\{x(t);\ t \in I\}$ if

$$M = N \quad \text{and} \quad dF^x_{z_n} = dF^y_{z_n} \quad \text{for every} \quad n = \overline{1,N}. \tag{4}$$

With P_t we shall denote the projector from $\mathcal{H}(x)$ onto the subspace $\mathcal{H}_t(x),\ t \in I$. If $z \in \mathcal{H}(x)$ then $\{z(t) = P_t z;\ t \in I\}$ is a process with orthogonal increments (see [**1**]).

II. In this part as was said at the beginning of the article, we shall study the equality problem for spectral types of the processes $\{x(t);\ t \in I\}$ and $\{y(t) = x(\varphi(t));\ t \in I\}$ for the case where the continuous nondecreasing function φ maps the interval I onto itself.

First we shall discuss processes with orthogonal increments.

Let $t \to F_{z_n}(t) = \|z_n(t)\|^2,\ t \in I$, be the distribution function of a stochastic process $\{z(t);\ t \in I\}$ with orthogonal increments, so that this function induces the measure which belongs to its maximal type dF_z (the multiplicity of this process in $N = 1$).

Let's define a new stochastic process $\{y(t),\ t \in I\}$ with the equality:

$$y(t) = (z \circ \varphi)(t) = z(\varphi(t)), \quad t \in I,$$

where $\varphi : I \to I$ is a nonrandom continuous nondecreasing function. It is not hard to show that $\{y(t); t \in I\}$ is also a stochastic process with othogonal increments. The distribution function of this process has the form:

$$F_y(t) = \|y(t)\|^2 = \|z(\varphi(t)\|^2 = F_x(\varphi(t)), \quad t \in I.$$

From this equality it follows:

$$m_{F_y}(B) = \int\limits_B dF_y(t) = \int\limits_B dF_x(\varphi(t)) = m_{F_y}(\varphi(B))$$

for any Borel set $B \subset I$.

It follows from theorem 1 given in [**7**] that the distribution functions $t \to F_z(t)$ and $t \to F_y(t),\ t \in I$ generate equivalent measures if and only if continous nondecreasing functions φ and $\varphi^{-1} : I \to I$ have the $N_{m_{F_z}}$ property, that is if $m_{F_z}(E) = 0$ implies $m_{F_z}(\varphi(E)) = m_{F_z}(\varphi^{-1}(E)) = 0$ for every m_{F_z} negligible set $E \in \mathcal{B}_I$.

So, the stochastic processes $\{z(t); t \in I\}$ and $\{y(t) = z(\varphi(t)),\ t \in I\}$ with orthogonal increments wil have the same spectral type if and only if φ and φ^{-1} are continuous nondecreasing $N_{m_{F_z}}$ functions.

Example 1. Let $\{W(t);\ t \in [0,1]\}$ be a Wiener process. Measures that belong to its spectral type dF_w are equivalent to the Lebesgue measure. Let $\varphi : [0,1] \to [0,1]$ be a nondecreasing and continuous function. Let's form a new process $\{z(t) = W(\varphi(t));\ t \in [0,1]\}$. The distribution function of this process has the form

$$F_z(t) = \varphi(t).$$

a) If $\varphi' > 0$ almost everywhere with respect to Lebesgue measure, we have $dF_z = dF_w$, because the continuity of φ and the condition $\varphi' > 0$ a.e. imply directly that φ and φ^{-1} are N functions, and this implies the equivalence $m_\varphi \sim m$.

(b) If $\varphi(t) = \begin{cases} 2t, & 0 \le t < 1/2 \\ 1\ , & 1/2 \le t \le 1. \end{cases}$, then $dF_z < dF_w$ because $m_\varphi < m$.

c) If $\varphi' = 0$ almost everywhere with respect to Lebesgue measure, then dF_w and dF_z are not comparable, because m_φ is a singular measure with respect to Lebesgue measure.

Let us now consider arbitrary stochastic processes with multiplicity $N = 1$.

THEOREM 1. *Let*

$$x(t) = \int_{t_0}^{t} g(t,u)\, dz(u), \quad t \in I, \tag{5}$$

be a proper canonical representation of the stochastic process $\{x(t);\ t \in I\}$ *with spectral type* dF^x *and let* m_{F_z} *be the measure in this spectral type induced by the distribution function* $t \to F_z(t) = \|z(t)\|^2,\ t \in I$. *Further, let the stochastic process* $\{y(t);\ t \in I\}$ *be defined by the equality*

$$y(t) = (x \circ \varphi)(t), \quad t \in I. \tag{6}$$

If φ *is a nonrandom continuous nondecreasing function mapping* I *onto* I *and such that* φ *and* φ^{-1} *have the* $N_{m_{F_z}}$ *property, we conclude*

(i) $dF^y = dF^x$;

(ii) *a proper canonical representation of the process* $\{y(t);\ t \in I\}$ *is given by*

$$y(t) = \int_{t_0}^{t} g^{(1)}(t,v)\, dz^1(v), \quad t \in I, \tag{7}$$

where $g^{(1)}(t,v) = g(\varphi(t),\ \varphi(v))$ *and* $z^1(v) = z(\varphi(v)),\ t_0 \le v \le t < \infty$.

Proof. It follows from formulae (5) and (6) that

$$y(t) = x(\varphi(t)) = \int_{t_0}^{\varphi(t)} g(\varphi(t)u)\, dz\,(u), \quad t \in I,$$

$$\|y(t)\|^2 = \int_{t_0}^{\varphi(t)} |g(\varphi(t),\, u)|^2 dF_z(u), \quad t \in I.$$

Since φ is a N_{mF_z} function, we may make the substitution $u = \varphi(v)$ (see [**2**] §20). We get the equality:

$$\|y(t)\|^2 = \int_{t_0}^{t} |g(\varphi(t),\ \varphi(v))|^2\, dF_z(\varphi(v)), \quad t \in I.$$

Since the distribution function of the process $\{z^1(t) = z(\varphi(t));\ t \in I\}$ with orthogonal increments has the form:

$$F_{z^1}(t) = \|z^1(t)\|^2 = \|z(\varphi(t))\|^2 = F_z(\varphi(t)),$$

we have

(8) $$m_{F_{z^1}}(B) = m_{F_z}(\varphi(B)),$$

for every Borel set $B \in \mathcal{B}_I$.

(9) $$m_{F_{z^1}} \sim m_{F_z},$$

because φ and φ^{-1} are $N_{m_{F_z}}$ function by assumption. Now we have to prove that (7) is a proper canonical representation of the process $\{y(t);\, t \in I\}$.

That is, have to show that for each $s \in I$ the equality:

(10) $$\int_{s_0}^{s_1} g^{(1)}(s_1, v)\, \overline{f(v)} dF_{z^1}(v) = 0 \quad \text{for every} \quad s_1 \in [t_0, s]$$

is possible only if

(11) $$\int_{s_0}^{s} |f(v)|^2\, dF_{z^1}(v) = 0.$$

It follows from (9) that $N_{m_{F_z}}$ function φ^{-1} is also a $N_{m_{F_{z^1}}}$ function.

Hence we can make the substitution $v = \varphi^{-1}(u)$ in the integral (10):

$$\int_{t_0}^{\varphi(s_1)} g^{(1)}(s_1, \varphi^{-1}(u))\overline{f(\varphi^{-1}(u))} dF_{z^1}(\varphi^{-1}(u)) = 0 \quad \text{for every}$$

$\varphi(s_1) \in [t_0,\, \varphi(s)]$.

If we now take $\varphi(s) = t$ and $\varphi(s_1) = t_1$ the last equality has the form (according to already introduced notation):

$$\int_{t_0}^{t_1} g(t_1, u)\overline{f(\varphi^{-1}(u))}\, dF_z(u) = 0, \quad \text{for every} \quad t_1 \in [t_0, t].$$

But this is possible only if

$$\int_{t_0}^{t} |f(\varphi^{-1}(u))|^2 \, dF_z(u) = 0, \tag{12}$$

because the family $\{g(t_1, u);\, u \in [t_0, t]$, the parameter $t \in I\}$ is complete in the space $\mathcal{L}_2(dF_z; t)$. With the substitution $u = \varphi(v)$ (12) gives (11).

Hence the representation (7) is proper canonical. It follows that the distribution function $t \to F_{z^1}(t)$ induces a measure of the maximal spectral type of the process $\{y(t); t \in I\}$. Since $m_{F_{z^1}} \sim m_{F_z}$ and m_{F_z} belong to the maximal spectral type of the process $\{x(t);\ t \in I\}$ we have $dF^x = dF^y$. Q.E.D.

Example 2. The representation

$$x(t) = \int_0^t (2t - u) \, dw(u), \quad t \in [0, 1],$$

is a proper canonical representation of the process $\{x(t);\ t \in [0, 1]\}$ with spectral type $dF^x : dF_w = dt$ (see [**3**])

If : $[0, 1] \to [0, 1]$ is such a continuous function that $\varphi' > 0$ almost everywhere with respect to Lebesgue measure on $[0, 1]$, we have with:

$$y(t) = \int_0^t (2\varphi(t) - \varphi(v)) \, dw(\varphi(v)), \quad t \in [0, 1]$$

a proper canonical representation of a process $\{y(t) = x(\varphi(t));\ t \in [0, 1]\}$ and $dF^y = dF^x$.

This is true by theorem 1, since it is not difficukt ti show, that φ and φ^{-1} are N functions.

Now we shall discuss an arbitrary stochastic process $\{x(t);\ t \in I\}$ with final multiplicity. Let Cramer's representation of this process be given by (1), and its spectral type by (2). If $y(t) = x(\varphi(t))$ the following claim holds.

THEOREM 2. *Stochastic processes* $\{x(t); t \in I\}$ *and* $\{y(t); t \in I\}$ *will have equal spectral types* $dF^x = dF^y$ *if the continuous and nondecreasing function* φ *which maps* I *onto* I *is such that for every* $n = \overline{1, N}$ *and for any* $m_{F_{z_n}}$ *—negligible set* $E \in \mathcal{B}_I$ *the relation*

$$m_{F_{z_n}}(E) = 0 \quad m_{F_{z_n}}(\varphi(E)) = m_{F_{z_n}}(\varphi^{-1}(E)) = 0 \tag{13}$$

holds. Cramer's representation of the process $\{y(t); t \in I\}$ *is given by the equality:*

$$y(t) = \sum_{n=1}^{N} \int_{t_0}^{t} g_n^{(1)}(t, u) \, dz_n^1(u), \quad t \in I, \quad 1 \le N < \infty, \tag{14}$$

where $\{(g_n^{(1)}(t,u))_{n=\overline{1,N}} = (g_n(\varphi(t)),\varphi(u))_{n=\overline{1,N}};\ u \in [t_0,t]\}$ *is a family of functions, the parameter* $t \in I$*, complete in the space* $\mathcal{L}_2(dF_{z^1};t)$*,* $(\{z_n^1(t) = z_n(\varphi(t)),$ $t \in I\})_{n=\overline{1,N}}$ *is a sequence of mutually orthogonal processes with ortogonal increments and finally*

$$t \to F_{z_1^1}(t) = \|z_1^1(t)\|^2, \dots, t \to F_{z_N^1}(t) = \|z_N^1(t)\|^2, \quad t \in I, \tag{15}$$

are the distribution functions inducing the measures belonging to the spectral type

$$dF^y : dF_{z_1^1} > dF_{z_2^1} > \cdots > dF_{z_N^1}, \quad 1 \le N < \infty. \tag{16}$$

Proof. Since:

$$y(t) = x(\varphi(t)) = \sum_{n=1}^{N} \int_{t_0}^{\varphi(t)} g_n(\varphi(t),u)\, dz_n(u), \quad t \in I,$$

it is:

$$\|y(t)\|^2 = \sum_{n=1}^{N} \int_{t_0}^{\varphi(t)} |g_n(\varphi(t),u|^2\, dF_{z_n}(u), \quad t \in I.$$

If we make the substitution $u = \varphi(v)$, what is possible since φ is a $N_{m_{F_{z_n}}}$ function for each $n = \overline{1,N}$ (see (13)) we get the equation

$$\|y(t)\|^2 = \sum_{n=1}^{N} \int_{t_0}^{t} |g_n(\varphi(t),\, \varphi(v))|^2\, dF_{z_n}(\varphi(v)), \quad t \in I.$$

Since the distribution functions of the processes $\{z_n^1(t) = z(\varphi(t));\, t \in I\}$, $n = \overline{1,N}$, with orthogonal increments have the form:

$$F_{z_n^1} = \|z_n^1(t)\|^2 = \|z_n(\varphi(t))\|^2 = F_{z_n}(\varphi(t)), \quad n = \overline{1,N},$$

it holds:

$$m_{F_{z_n^1}}(B) = m_{F_{z_n}}(\varphi(B))$$

for every Borel set $B \in \mathcal{B}_I$. Hence:

$$m_{F_{z_n}} \sim m_{F_{z_n^1}} \quad \text{for every} \quad n = \overline{1,N},$$

because φ and φ^{-1} are by assumption of the theorem $N_{m_{F_{z_n}}}$ functions for every $n = \overline{1,N}$.

To prove that (14) represents the proper canonical representation of the process $\left\{y(t) = \sum\limits_{n=1}^{N} y_n(t); t \in I\right\}$, on the bases propositiion 2 in [**10**] it is enough to show that $y_n(t) \in \mathcal{H}_t(y)$ for every $t \in I$ and $n = \overline{1,N}$. This is true because it follows from theorem 1, that

$$y_n(t) = \int_{t_0}^{t} g_N^{(1)}(t,u)\, dz_n^1(u), \quad t \in I, \quad n = \overline{1,N},$$

is a proper canonical representation of the process $\{y_n(t);\, t \in I\}$.

From the proper canonical representation (1) of the process $\left\{x(t) = \sum_{n=1}^{N} x_n(t);\; t \in I\right\}$ we see, that $x_n(t) \in \mathcal{H}_t(x)$ for every $t \in I$ and $n = \overline{1,N}$. Since for every $t \in I$ there exists $s \in I$ such that $t = y(s)$ and since it is obvious that $\mathcal{H}_{\varphi(s)}(x) = \mathcal{H}_s(y)$, it is

$$y_n(s) = x_n(\varphi(s)) \in \mathcal{H}_{\varphi(s)}(x) = \mathcal{H}_s(y) \quad \text{for every} \quad s \in I, \quad n = \overline{1,N}.$$

So the representation (14) is proper canonical. Hence the distribution functions $t \to F_{z_n}(t)$, $t \in I$, $n = \overline{1,N}$ induce measures which belong to maximal spectral types $dF_{z_n^1}$ of the process $\{y(t);\, t \in I\}$. Since $m_{F_{z_n}} \sim m_{F_{z_n^1}}$ for every $n = \overline{1,N}$, and since $m_{F_{z_n}}$, $n = \overline{1,N}$, belong to maximal spectral type dF_{z_n} of the process $\{x(t); t \in I\}$, we have $dF^z = dF^y$. Q.E.D.

Remark. Altough the assumption, that the multiplicity N of the process $\{x(t);\, t \in I\}$ is finite was not explicitely used in the proof of the theorem, the assumption is necessary for the following reason: when $N = \infty$ it can happen, that there is no other function φ but the identity $\varphi(t) = t$, $t \in I$, which satisfies all the suppositions of the theorem. For example, if the process $\{x(t);\, t \in I\}$ has a discrete spectral type, to which the measure of example 8 from [**7**] belong then, as is shown in this example it has to be $\varphi(t) = t$, $t \in I$. Hence every transformation $\varphi(t) \neq t$ would change the spectral type of this particular process.

Example 3. Let $h : [0,1] \to R$ be an absolutely continuous function with strictly positive Radon-Nicodym derivative which belongs to $\mathcal{L}_1([0,t];\, dt)$ for every $t \in [0,1]$ and which does not belong to $\mathcal{L}_2(\langle\alpha,\beta\rangle;\, dt)$ for any interval $\langle\alpha,\beta\rangle \subset [0,1]$. Then, as is shown in [**4**] example 3, stochastic process:

$$x(t) = W_1(t) + h(t)W_2(t) + h^2(t)W_3(t) + \cdots + h^{N-1}(t)W_N(t), \\ t \in [0,1], \quad N < \infty,$$

where $(\{W_n(t);\, t \in [0,1]\})_{n=\overline{1,N}}$ is a sequence of mutually orthogonal Wiener processes, has the spectral type

$$dF^x : \underbrace{dt \geq dt \geq \cdots \geq dt}_{N\text{—times}}.$$

If $F:[0,1]\to]0,1]$ is a continuous function such that $F'>0$ almost everywhere with respect to Lebesgue measure then F and F^{-1} are N functions, what is not difficult to show. Hence the stochastic process:

$$y(t)=x(F(t))=z_1(t)+g(t)z_2(t)+g^2(t)z_3(t)+\cdots+g^{N-1}(t)z_N(t),$$

$t\in[0,1]$, $N<\infty$, where $z_n(t)=W_n(F(t))$, $n=\overline{1,N}$, $g(t)=h(F(t))$, has spectral type of the form:

$$dF^y:\underbrace{dF\geq dF\geq\cdots\geq dF}_{N\text{—times}}.$$

and $dF^x=dF^y$, follows from theorem 2, since the function F satisfies the conditions of that theorem.

Now we shall construct a function $h:[0,1]\to R$ with properties mentioned above. Let:

$$f_n(t)=\begin{cases}0 & \text{for}\quad t=0\\ \dfrac{1}{n^2\sqrt{t}} & \text{for}\quad 0<t\leq\dfrac{1}{n}\\ \dfrac{1}{n^2\sqrt{t-\frac{1}{n}}} & \text{for}\quad \dfrac{1}{n}<t\leq\dfrac{2}{n}\\ \dots\dots\dots\dots\dots\dots & \\ \dfrac{1}{n^2\sqrt{t-\frac{n-1}{n}}} & \text{for}\quad \dfrac{n-1}{n}<t\leq 1\end{cases}$$

be the n-th term of the sequence $(f_n:[0,1]\to R_{n=\overline{1,\infty}}$

Since: $\displaystyle\int\limits_0^1 f_n(t)\,dt=\sum_{k=0}^{n-1}\int\limits_{k/n}^{(k+1)/n}\frac{1}{n^2\sqrt{t-\frac{k}{n}}}\,dt=\sum_{k=0}^{n-1}\frac{2\sqrt{\frac{1}{n}}}{n^2}=\frac{2}{n^{5/2}}$, the series

$$\sum_{n=2}^{\infty}\int\limits_0^1 f_n(t)\,dt=\sum_{n=1}^{\infty}\frac{2}{n^{5/2}}$$

converges. Also, since $(f_n)_{n=\overline{1,\infty}}$ is sequence of nonnegative measurable functions on the segment $[0,1]$ the series $\sum\limits_{n=1}^{\infty}f_k(t)$ converges almost everywhere on $[0,1]$ to an integrable function $f:[0,1]\to R$ and

$$\int\limits_0^1 f(t)\,dt=\sum_{n=1}^{\infty}\int\limits_0^1 f_n(t)\,dt,$$

([**6**], VII. § 1, T 11). Let now:

$$h(t)=\int\limits_0^1 f(u)\,dy=\sum_{n=2}^{\infty}\int\limits_0^1 f_n(u)\,dy,\quad t\in[0,1].$$

It is obvious that $h:[0,1]\to R$ is an absolutely continuous function with positive Random-Nicodym derivative, which belongs to the space $\mathcal{L}_1([0,t];\,dt)$ for each $t\in[0,1]$. Let's show that this derivative doesn't belong to the space $\mathcal{L}_2(\langle\alpha,\beta\rangle;\,dt)$ that is $\int\limits_\alpha^\beta |h^I(t)|^2\,dt=+\infty$ for arbitrary interval $\langle\alpha,\beta\rangle\subset[0,1]$. For any interval $\langle\alpha,\beta\rangle\subset[0,1]$ we can find such $m\in N$ that for at least one $k\in N$, $0\le k\le m-1$, $\left\langle\frac{k}{m},\frac{k+1}{m}\right\rangle\subset\langle\alpha,\beta\rangle$.

Since it is:

$$\int\limits_{k/m}^{(k+1)/m}\frac{du}{m^4\left(u-\frac{k}{m}\right)}=\frac{1}{m^4}\lim_{A\to\frac{k}{m}}\int\limits_A^{\frac{k+1}{m}}\frac{du}{u-\frac{k}{m}}=+\infty$$

we have

$$\int\limits_\alpha^\beta |h^I(t)|^2\,dt=\int\limits_\alpha^\beta f^2(t)\,dt=\int\limits_\alpha^\beta\left(\sum_{n=2}^\infty f_n(t)\right)dt\ge\int\limits_\alpha^\beta\left(\sum_{n=2}^\infty f_n^2(t)\right)dt$$

$$\ge\int\limits_\alpha^\beta\left(\sum_{n=2}^m f_n^2(t)\right)dt=\sum_{n=2}^m\int\limits_\alpha^\beta f_n^2(t)\,dt\ge\int\limits_{k/m}^{\frac{k+2}{m}}\frac{dt}{m^4\left(t-\frac{k}{m}\right)}=+\infty,$$

what we had to show.

III. Now we shall discuss the problem of invariance of spectral type of the process $\{x(t):\,t\in I\}$ with respect to a transformation φ, which maps I_1 onto I.

Since in the case when $I_1\ne I$ the spaces on which measures belonging to spectral types dF^x and dF^y are defined are not the same , we shall use the following definition, which allows us to compare them.

Definition. *Let:*

$$dF^x: dF_1^x>dF_2^x>\cdots>dF_N^x,\qquad 1\le N\le\infty,$$
$$dF^y: dF_1^y>dF_2^y>\cdots>dF_M^y,\qquad 1\le M\le\infty,$$

be spectral types of stochastic processes $\{x(t);\,t\in I\}$, $\{y(t);\,t\in I_1\}$ *and let* $m_{F_n^x}$, $n=\overline{1,N}$, $(m_{F_m^y}, m=\overline{1,M})$ *be the measures belonging to spectral types* dF_n^x, $n=\overline{1,M}$, $(dF_m^y,\, m=\overline{1,M})$. *Besides, let* φ *be a linear function, which maps* I *onto* I_1.

(a) *We shall say that the spectral type* dF_n^x *is* l*-subordinated (*l*-equivalent) to the spectral type* dF_n^y *if the measure* $m_{F_n^x}$ *is* l*-subordinated (*l*-equivalent) to the measure* $m_{F_n^y}$ *in the sense definition* 2 *in* [**7**].

(b) *For stohastic processes* $\{x(t);t\in I\}$ *and* $\{y(t);\,t\in I_1\}$ *we shall say that they have* l*-equivalent spectral types if* $M=N$ *and if for every* $n=\overline{1,N}$, dF_n^x *is* l*-equivalent to the spectral type* dF_n^y.

If $I_1 = I$ the definition of subordination and equality of spectral types of stochastic processes, given in part I is equivalent to the just cited definition, this follows from the definition 2 in [**7**] and the comment given there.

Since l-equivalence of spectral types is defined in terms of l-equivalence of measures belonging to them and since l-equivalence of measures is defined in terms of the "common" equivalence by introducing the function $\varphi_1 = \varphi\circ : I \to I$, which inherits all the properties of $\varphi : I_1 \to I$ that are important for the measurability of transformed sets (because l is an affine transformation it doesn't change the structure of sets with respect to L–S measures) we can conclude:

All the results that we get in case when φ maps I onto I are true in the case when φ maps I_1 onto I. Only the terminolgy is slightly changed. For example, theorem 2 would read like this:

Stohastic processes $\{x(t); t \in I\}$ and $\{y(t) = x(\varphi(t));\ t \in I_1\}$ have l-equivalent spectral types if the nonrandom continuous nondecreasing function $\varphi : I_1 \to I$ is such, that functions $\varphi_1 = \varphi \circ l$ and $\varphi^{-1} : I \to I$ have the $N_{m_{F_{z_n}}}$ — property for every $= \overline{1, N}$.

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