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INTEGRABILITY THEOREMS FOR TRIGONOMETRIC SERIES WITH POSITIVE COEFFICIENTS

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1. Introduction and results

Let g be an odd function and let f be an even function defined on $(0,\pi)$, periodic with period 2π, and let their Fourier series be

$$g(x)=\sum_1^\infty b_n \sin nx \qquad f(x)=\frac{1}{2}a_0+\sum_1^\infty a_n\cos nx$$

with $a_n\geq 0$, $b_n\geq 0$. In his book [**3**] Boas proved the following two theorems concerning integrability of g (similar statements hold for f).

Theorem A. Let $0<\gamma<1$. *Then*

$$\int_{\to 0}^{\pi} x^{-\gamma}g(x)dx<\infty \Leftrightarrow \sum_1^\infty n^{\gamma-1}b_n<\infty$$

$\Bigg($ *where* $\int_{\to 0}^{\pi}=\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\pi}$ *is the integral in Cauchy's sense* $\Bigg)$.

Theorem B. *Let* $1<\gamma<2$. *Then*

$$x^{-\gamma}g(x)\in L(0,\pi)\Leftrightarrow \sum_1^\infty n^{\gamma-1}b_n<\infty.$$

Izumi and Izumi [**4**] have proved a generalization of Theorem A in which the function $x^{-\gamma}$ is replaced by a monotone decreasing function $\xi(x)$, and Hasegawa [**5**] has proved a generalization of Theorem B in which $x^{-\gamma}$ is replaced by a function $\alpha(x)$ having the following properties: $x\alpha(x)$ is decreasing and $t^{-1}\int_t^\eta \alpha(x)dx\leq C\alpha(t)$, for some η, $0<\eta\leq\pi$ and all t, $0<t\leq\eta$.

In the present paper we prove another generalization of Theorem B (Theorem 1). We remark that no monotony condition iz required for the multiplier function K. This is a consequence of the following observations: different properties of the function $x^{-\gamma}$ are essential for Theorems A and B. While the monotony of $x^{-\gamma}$ plays a central role in the proof of Theorem A, in Theorem B no use is made of monotony, but only of the regular variation of $x^{-\gamma}$. Indeed, the main step in the proof of both theorems is the estimation of the integral $\int_0^{\pi} x^{-\gamma} \sin nx dx = n^{\gamma-1} \int_0^{n\pi} x^{-\gamma} \sin x dx$. For $1 < \gamma < 2$ this integral is absolutely convergent, and this is a consequence of the regular variation properties of $x^{-\gamma}$ only (cf. [**1**]). On the other hand, for $0 < \gamma < 1$, the integral converges nonabsolutely and the monotony of $x^{-\gamma}$ is essential for this statement.

For the definition and properties of 0-regularly varying ($0-RV$) functions (in the sense of Karamata) we refer to [**2**]. The symbol $\mathcal{K}(\underline{\varrho}, \overline{\varrho})$ denotes the class of all $0-RV$ functions with lower index $\underline{\varrho}$ and upper index $\overline{\varrho}$. The symbol $\asymp$ iz defined by: $f(x) \asymp g(x)$ on $[a, \infty)$ if there are two positive constants C_1, C_2 such that $0 < C_1 g(x) \le f(x) \le C_2 g(x) < \infty$. By the letter C, possibly with subscripts, we denote a positive constant, not necessarily the same at each appearence.

Theorem 1. *Let $b_n \ge 0$ and let g be defined by $g(x) = \sum_1^\infty b_n \sin nx$. Let K be a positive function defined on $(1/\pi, \infty)$ such that $xK\left(\frac{1}{x}\right) \in L(0,\pi)$ and*

$$\int_{1/\pi}^{u} t^{-2} K(t) dt \asymp u^{-1} K(u), \quad \textit{for } u > \frac{1}{u}. \tag{1}$$

Then

$$\sum_1^\infty n b_n \int_0^{1/n} x K\left(\frac{1}{x}\right) dx < \infty \Rightarrow g(x) K\left(\frac{1}{x}\right) \in L(0,\pi) \tag{a}$$

and conversely

$$g(x) K\left(\frac{1}{x}\right) \in L(0,\pi) \Rightarrow \sum_1^\infty b_n \frac{K(n)}{n} < \infty. \tag{b}$$

Theorem 2. *Let $a_n \ge 0$ and let f be defined by $f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nx$.*

Let K be a positive function defined on $(1/\pi, \infty)$ such that $x^2 K\left(\frac{1}{x}\right) \in L(0,\pi)$ and condition (1) is satisfied. Then

$$\sum_1^\infty n^2 a_n \int_0^{1/n} x^2 K\left(\frac{1}{x}\right) dx < \infty \ \Rightarrow \ (f(0) - f(x)) K\left(\frac{1}{x}\right) \in L(0,\pi) \tag{a}$$

and conversely

(b) $$(f(0)-f(x))K\left(\frac{1}{x}\right)\in L(0,\pi)\ \Rightarrow\ \sum_1^\infty a_n\frac{K(n)}{n}<\infty.$$

COROLLARY 1. *Let* $b_n\ge 0$ *and let* $g(x)=\sum\limits_1^\infty b_n\sin nx$. *Let* $K\in\mathcal{K}(1<\underline{\varrho},\overline{\varrho}<2)$. *Then*

$$g(x)K\left(\frac{1}{x}\right)\in L(0,\pi)\Leftrightarrow\ \sum_1^\infty b_n\frac{K(n)}{n}<\infty.$$

CORROLARY 2. *Let* $a_n\ge 0$ *and let* $f(x)=\frac{1}{2}a_0+\sum\limits_1^\infty a_n\cos nx$. *Let* $K\in\mathcal{K}(1\le\underline{\varrho},\ \overline{\varrho}<3)$. *Then*

$$(f(0)-f(x))K\left(\frac{1}{x}\right)\in L(0,\pi)\ \Leftrightarrow\ \sum_1^\infty a_n\frac{K(n)}{n}<\infty.$$

2. Some properties of 0-RV functions

Let us remark that condition (1) is a characteristic property for 0-regularly varying functions ([**2**]), i.e. a function K satisfying (1) is an $0-RV$ function with lower index $\underline{\varrho}>1: K\in\mathcal{K}(1<\underline{\varrho},\ \overline{\varrho})$. In the following lemma we list some properties of $0-RV$ functions which follow from (1).

LEMMA. *Let K be a positive function defined on* $(1/\pi,\infty)$ *such that* (1) *holds. Then*

1° $u^{-1}K(u)$ *is almost increasing* ($u^{-1}K(u)\nearrow\!\nearrow$), *i.e. there is a constant* $C\ge 0$ *such that* $u^{-1}K(u)\le Cv^{-1}$, *for* $1/\pi<u<v$.

2° *There is a* $\tau<1$ *such that* $u^{-\tau}K(u)$ *is almost decreasing* ($u^{-\tau}K(u)\searrow\!\searrow$), *i.e. there is a constant* $C\ge 0$ *such that* $u^{-\tau}K(u)\ge Cv^{-\tau}K(v)$, *for* $1/\pi<u<v$.

3° *If* $K\left(\frac{1}{x}\right)g(x)\in L(0,\pi)$, *then* $x^{-1}g(x)\in L(0,\pi)$.

4° *Let* $F(x)=\int\limits_0^x|f(t)|\,dt$. *Then*

$$\int_0^1 K\left(\frac{1}{x}\right)F(x)\,dx\le C\int_0^1 K\left(\frac{1}{x}\right)x|f(x)|\,dx$$

for some positive constant C.

5° *If* $c_n\downarrow 0$, *then the series* $\sum\limits_1^\infty c_n\frac{K(n)}{n}$ *and* $\sum\limits_1^\infty(c_n-c_{n+1})\frac{K(n)}{n}$ *are equiconvergent.*

Proof.

1° and 2° cf. [**2**].

3° It is easily seen that

$$\int_0^\pi x^{-1}|g(x)|\,dx = \int_0^\pi |g(x)|K\left(\frac{1}{x}\right)\left(xK\left(\frac{1}{x}\right)\right)^{-1} dx$$

$$\le \sup_{0<x<\pi}\left(xK\left(\frac{1}{x}\right)\right)^{-1}\int_0^\pi |g(x)|K\left(\frac{1}{x}\right)dx = C\sup_{1/\pi<1<\infty}(t^{-1}K(t))^{-1}$$

$$= C(\inf_{1\pi<t<\infty}(t^{-1}K(t))^{-1} = C_1\left(\pi K\left(\frac{1}{\pi}\right)\right)^{-1},$$

since $t^{-1}K(t)$ is almost increasing, by 1°.

4° We by changing the order of integration and by making use of (1)

$$\int_0^1 K\left(\frac{1}{x}\right)F(x)dx = \int_0^1 K\left(\frac{1}{x}\right)\int_0^x |f(t)|dt\,dx = \int_0^1 |f(t)|\int_t^1 K\left(\frac{1}{x}\right)dx\,dt =$$

$$= \int_0^1 |f(t)|\int_1^{1/t} K(u)u^{-2}du dt \le C\int_0^1 |f(t)|\left(\frac{1}{t}\right)^{-1}K\left(\frac{1}{t}\right)dt = C\int_0^1 |f(t)|tK\left(\frac{1}{t}\right)dt.$$

5° this is Lemma 2 of [**1**].

3. Proof of the Theorems

We prove Theorem 1 only, the proof of Theorem 2 being very similar.

a) First, by the definition of g, we have

(2)
$$\int_0^\pi K\left(\frac{1}{x}\right)|g(x)|\,dx = \int_0^\pi K\left(\frac{1}{x}\right)\Big|\sum_1^\infty b_n \sin nx\Big|\,dx \le \int_0^\pi K\left(\frac{1}{x}\right)\sum_1^\infty |b_n||\sin nx|dx$$

$$= \sum_1^\infty b_n\int_0^\pi K\left(\frac{1}{x}\right)|\sin nx|\,dx.$$

Next we prove that

$$\int_0^\pi K\left(\frac{1}{x}\right)|\sin nx|\,dx \le Cn\int_0^{1/n} xK\left(\frac{1}{x}\right)dx. \tag{3}$$

Really,

$$\int_0^\pi K\left(\frac{1}{x}\right)|\sin nx|\,dx = \int_0^{1/n} + \int_{1/n}^\pi = I_1 + I_2.$$

Now for the first integral we have obviously

$$I_1 = \int_0^{1/n} K\left(\frac{1}{x}\right)|\sin nx|\,dx \le n\int_0^{1/n} xK\left(\frac{1}{x}\right)dx \tag{4}$$

and for the second

$$I_2 = \int_{1/n}^\pi K\left(\frac{1}{x}\right)|\sin nx|\,dx \le \int_{1/n}^\pi K\left(\frac{1}{x}\right)dx = \int_{1/\pi}^n K(t)t^{-2}\,dt \le Cn^{-1}K(n), \tag{5}$$

since K satisfies (1).

Now, by Lemma 1°, the function $u^{-1}K(u)$ is almost increasing, thus

$$n\int_0^{1/n} xK\left(\frac{1}{x}\right)dx = n\int_n^\infty t^{-1}K(t)t^{-2}\,dt \ge Cn\cdot n^{-1}K(n)\int_n^\infty t^{-2}\,dt = \tag{6}$$
$$= CK(n)\cdot n^{-1}.$$

From (5) and (6) it follows that

$$I_2 \le Cn\int_0^{1/n} xK\left(\frac{1}{x}\right)dx$$

which, together with (4), proves (3).

Finally, by substituting (3) into (2), we obtain

$$\int_0^\pi K\left(\frac{1}{x}\right)|g(x)|\,dx \le C\sum_0^\infty |b_n|n\int_0^{1/n} xK\left(\frac{1}{x}\right)dx$$

which proves part a) of the Theorem.

b) Assume that $g(x)K\left(\frac{1}{x}\right) \in L(0,\pi)$. By Lemma 3° it follows that $x^{-1}g(x) \in L(0,\pi)$; hence by Theorem 4.1. [**3**] the seies $\sum_1^\infty b_n$ is convergent.

Denote $B_k = \sum_{j=0}^\infty b_{2j+k+1}$. Then B_k decreases to zero and we obtain by partial summation (cf. Lemma 2.2. [**3**])

$$g(x) = \sum_1^\infty b_n \sin nx = B_0 \sin x + 2\sum_1^\infty B_n \cos nx \sin x$$

or

$$h(x) = \frac{g(x)}{\sin\, x} = B_0 + 2\sum_1^{\infty} B_0 \cos\, nx.$$

Denote $\varphi(x) = h(x) - B_0 = 2\sum\limits_1^{\infty} B_n \cos\, nx$. It is easily seen that

$$\varphi(x)xK\left(\frac{1}{x}\right) \in L(0,\pi). \tag{7}$$

Indeed,

$$\int_0^{\pi} |\varphi(x)xK\left(\frac{1}{x}\right) dx \leq \int_0^{\pi} |h(x)xK\left(\frac{1}{x}\right) dx + B_0 \int_0^{\pi} xK\left(\frac{1}{x}\right) =$$

$$= \int_0^{\pi} |g(x)| \frac{x}{\sin\, x} K\left(\frac{1}{x}\right) dx + B_0 \int_0^{\pi} xK\left(\frac{1}{x}\right) dx$$

and both integrals converge by the assumptions of the theorem. Since φ is integrable and B_n monotone, it follows that B_n are the Fourier coefficients of φ and we can put

$$\Phi(x) = \int_0^x \varphi(t)dt = 2\sum_1^{\infty} \frac{B_k}{k} \sin\, kx$$

$$F(x) = \int_0^x \Phi(t)\, dt = 2\sum_1^{\infty} \frac{B_k}{k^2}(1 - \cos\, kx) = 4\sum_1^{\infty} \frac{B_k}{k^2} \sin^2 \frac{kx}{2}.$$

Next we prove that

$$F\left(\frac{1}{n+1}\right) \geq C\frac{B_n}{n} \tag{8}$$

Indeed,

$$F\left(\frac{1}{n+1}\right) = 4\sum_{k=1}^{\infty} \frac{B_k}{k^2} \sin^2 \frac{k}{2(n+1)} \geq 4\sum_{k=1}^{\infty} \frac{B_k}{k^2} \sin^2 \frac{k}{2(h+1)}$$

$$4\left(\frac{2}{\pi}\right)^2 \sum_{k=1}^{n} \left(\frac{k}{2(n+1)}\right)^2 = \frac{4}{\pi^2} \frac{1}{(n+1)^2} \sum_{k=1}^{n} B_k \geq \frac{4}{\pi^2} \frac{1}{n} B_n \cdot n = \frac{4}{\pi^2} \frac{B_n}{n}$$

where we have used that $\sin \frac{k}{2(n+1)} \geq \frac{2}{\pi} \frac{k}{2(n+1)}$, for $\frac{k}{2(n+1)} < \frac{1}{2}$ and that B_n is a monotone decreasing sequence.

Next, put $\Phi^+(x) \int\limits_0^x |\varphi(t)|\, dt$ and $F^+(x) = \int\limits_0^x |\Phi(t)|\, dt$. The functions Φ^+ and F^+ are positive and increasing and

$$|F(x)| \leq F^+(x) \leq \int_0^x \Phi^+(t)\, dt \leq x\Phi^+(x). \tag{9}$$

Therefore, by (8) and (9) it follows that

$$\frac{B_n}{n} \le C\frac{1}{n+1}\Phi^+\left(\frac{1}{n+1}\right). \tag{10}$$

Now we are ready to prove

$$\sum_1^\infty \frac{B_n K(n)}{n^2} \le \int_0^1 |\varphi(x)| x K\left(\frac{1}{x}\right), dx \tag{11}$$

The proof runs as follows

$$\begin{aligned}
\sum_1^\infty \frac{B_n}{n}\frac{K(n)}{n} &\le C\sum_1^\infty \frac{1}{n+1}\Phi^+\left(\frac{1}{n+1}\right)\frac{K(n)}{n} \qquad [\text{by } (10)]\\
&\le C\sum_1^\infty \frac{1}{n+1}\Phi^+\left(\frac{1}{n+1}\right)\int_n^{n+1}\frac{K(t)}{t}\,dt \qquad [\text{since } u^{-1}K(u) \nearrow\!\!\nearrow]\\
&= C\sum_1^\infty \frac{1}{n+1}\Phi^+\left(\frac{1}{n+1}\right)\int_{1/(n+1)}^{1/n} xK\left(\frac{1}{x}\right)x^{-2}\,dx\\
&\le C\sum_1^\infty \int_{1/(n+1)}^{1/n} x\Phi^+(x)K\left(\frac{1}{x}\right)x^{-1}\,dx \quad [\text{since } x\Phi^+(x)\uparrow]\\
&= C\int_0^1 K\left(\frac{1}{x}\right)\Phi^+(x)\,dx
\end{aligned}$$

Applying Lemma 4° to the last integral we obtain (11). Thus, by (7) if follows that the series $\sum\limits_1^\infty B_n\frac{K(n)}{n^2}$ is convergent. To complete the proof of the theorem we have to prove that this implies that the series $\sum\limits_1^\infty b_n\frac{K(n)}{n}$ is convergent. First, by Lemma 5° it follows that

$$\sum_1^\infty (B_n - B_{n+1})\frac{K(n)}{n} < \infty \tag{12}$$

and since $u^{-1}K(u)$ is almost increasing

$$\begin{aligned}
\sum_1^\infty (B_{n+1} - B_{n+2})\frac{K(n)}{n} &\le C\sum_1^\infty (B_{n+1} - B_{n+2})\frac{K(n+1)}{n+1}\\
&= C\sum_2^\infty (B_n - B_{n+1})\frac{K(n)}{n}.
\end{aligned} \tag{13}$$

Thus from (12) and (13) the following series

$$\sum_1^\infty (B_n - B_{n+2}) \frac{K(n)}{n} = \sum_1^\infty b_{n+1} \frac{K(n)}{n} \tag{14}$$

is also convergent. Now, since by Lemma 2° there is a $\tau > 1$ such that $u^{-\tau}K(u)$ is almost decreasing, we have

$$\frac{K(n+1)}{n+1} = (n+1)^{-\tau}K(n+1)(n+1)^{\tau-1} \le Cn^{-\tau}K(n)C_1 n^{\tau-1} = C_2 n^{-1}K(n).$$

Whence

$$\sum_1^\infty b_{n+1} \frac{K(n+1)}{n+1} \le C_2 \sum_1^\infty b_{n+1} \frac{K(n)}{n}$$

and the last series is convergent, by (14). Thus we have proved that $\sum\limits_1^\infty b_n \frac{K(n)}{n}$ is convergent, which completes the proof of the theorem.

Proof of Corollary 1.

Let $K \in \mathcal{K}(1 < \underline{\varrho},\ \overline{\varrho} < 2)$. The assumption $\underline{\varrho} > 1$ implies condition (1) of Theorem 1. On the other hand, from the assumption $\overline{\varrho} < 2$ it follows that

$$\int_u^\infty t^{-3} K(t)\, dt \asymp u^{-2} K(u).$$

Thus

$$n \int_0^{1/n} x K\left(\frac{1}{x}\right) dx = n \int_n^\infty t^{-1} K(t) t^{-2}\, dt \asymp n\, n^{-2} K(n) = n^{-1} K(n)$$

which means that the series

$$\sum_1^\infty n b_n \int_0^{1/n} x K\left(\frac{1}{x}\right) dx \quad \text{and} \quad \sum_1^\infty b_n \frac{K(n)}{n}$$

are equiconvergent. Now Corollary 1 follows from Theorem 1.

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