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ON SOME MERCERIAN THEOREMS

Dedicated to profesor Đuro Kurepa in honor of hih 75-th birtday

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In this paper we shall prove two Mercerian theorems involving the generalized arithmetical means, which contain, as particular cases, some known theorems of the same kind.

The first theorem is

THEOREM 1. *Let* $p_k \geq 0$ *for all* k, *and* $0 < P_n = \sum_{k=1}^{n} p_k \to \infty$, $n \to +\infty$. *Then for any sequence* $\{s_n\}$, *from*

$$\frac{p_n}{p_n + q_n} s_n + \frac{q_n}{p_n + q_n} \frac{1}{P_n} \sum_{k=1}^{n} p_k s_k \to s, \quad n \to \infty, \tag{1}$$

follows

$$\frac{1}{P_n} \sum_{k=1}^{n} p_k s_k \to s, \quad n \to \infty, \tag{2}$$

whenever the sequence $\{q_n\}$ *satisfies the conditions*

$$P_n |d_n| \to \infty \ \ \text{and} \ \ \sum_{k=1}^{n} |d_{k-1}| |p_k + q_k| = 0(P_n |d_n|), \quad n \to \infty \tag{3}$$

where

$$d_n = \prod_{k=1}^{n} \Big(1 + \frac{q_k}{P_k}\Big), \quad d_0 = 0. \tag{4}$$

and $q_n \notin \{-p_n, -P_n\}$.

Proof. In his paper [**1**] Karamata gives the following generalization of the theorem of Stolz:

Theorem K. *Let* $\{y_n\}$ *be a sequence of real number such that* $y_n - y_{n-1} \neq 0$ *for all* n,

$$|y_n| \to \infty \quad \text{and} \quad \sum_{k=1}^{n} |y_k - y_{k-1}| = O(|y_n|), \quad n \to \infty. \tag{5}$$

Then, from

$$\frac{x_n - x_{n-1}}{y_n - y_{n-1}} \to \alpha, \quad n \to \infty, \tag{6}$$

follows

$$\frac{x_n}{y_n} \to \alpha, \quad n \to \infty. \tag{7}$$

Set now in this theorem $x_n = d_n \sum\limits_{k=1}^{n} p_k s_k$ and $y_n = d_n P_n$. Taking into account that

$$P_n(d_n - d_{n-1}) = P_n d_{n-1}\left\{1 + \frac{q_n}{P_n} - 1\right\} = d_{n-1} q_n,$$

we have

$$\begin{aligned}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} &= \frac{d_n \sum\limits_{k=1}^{n} p_k s_k - d_{n-1} \sum\limits_{k=1}^{n-1} p_k s_k}{d_n P_n - d_{n-1} P_{n-1}} = \\ &= \frac{d_{n-1}\left\{\left(1 + \frac{q_n}{P_n}\right) \sum\limits_{k=1}^{n} p_k s_k - \sum\limits_{k=1}^{n-1} p_k s_k\right\}}{d_{n-1}\left\{P_n\left(1 + \frac{q_n}{P_n}\right) - P_{n-1}\right\}} = \\ &= \frac{p_n s_n + \frac{q_n}{P_n} \sum\limits_{k=1}^{n} p_k s_k}{p_n + q_n}.\end{aligned}$$

From that and Theorem K our Theorem 1. follows in a few lines.

In the particular case $p_k = 1$, $P_n = n$ we obtain the following theorem of J. Karamata ([**1**], p. 24):

Corollary 1.1. *From*

$$s_n + q_n \frac{1}{n} \sum_{k=1}^{n} s_k \sim (1 + q_n) s, \quad n \to \infty, \tag{8}$$

follows $\frac{1}{n} \sum\limits_{k=1}^{n} s_k \to s$, $n \to \infty$, *whenever the sequence* $\{q_n\}$ *satisfies the conditions*

$$n|d_n| \to \infty \quad \text{and} \quad \sum_{k=1}^{n} |d_{k-1}||1 + q_k| = O(n|d_n|), \quad n \to \infty,$$

where $d_n = \prod_{k=1}^{n}\left(1 + \frac{q_k}{k}\right)$.

Suppose now that $\underline{\lim}\, q_n \geq 1$ and that $\sum_{k=1}^{\infty} \frac{1}{P_k}$ is a divergent series. Without loss of generality we can suppose that $q_n \geq q > 1$ for all n; thus d_n, defined by (4), will be increasing with limit ∞. Then

$$\begin{aligned}\sum_{k=1}^{n} |d_{k-1}||p_k + q_k| &= \sum_{k=1}^{n} d_{k-1}p_k + \sum_{k=1}^{n} d_{k-1}q_k = \\ &= \sum_{k=1}^{n} d_{k-1}p_k + \sum_{k=1}^{n} P_k(d_k - d_{k-1}) = \\ &= \sum_{k=1}^{n} d_{k-1}p_k + \sum_{k=1}^{n} P_k d_k - \sum_{k=1}^{n-1} P_{k+1}d_k = \\ &= \sum_{k=1}^{n} d_{k-1}p_k + P_n d_n + \sum_{k=1}^{n-1}(P_k - P_{k+1})d_k = P_n d_n,\end{aligned}$$

which shows that all the conditions of Theorem 1. are satisfied. So, we obtain

THEOREM 2. *From* (1) *and* $\underline{\lim}\, q_n > 1$ *follows* (2) *for any sequence* $\{s_n\}$.

With $p_k = 1$ for all k, we obtain from Theorem 2. a theorem of Vijayaraghavan [**2**]:

COROLLARY 2.1. *From* $\underline{\lim}\, q_n > 1$ *and* (8) *follows*

$$\frac{1}{n}\sum_{k=1}^{n} s_k \to s, \quad n \to \infty.$$

REFERENCES

[1] J. Karamata, *Sur quelques inversions d'une proposition de Cauchy et leurs généralizations*, Tôhoku Math. J. **36** (1933), 22–28.

[2] T. Vijayaraghavan: *A Generalization of the Theorem of Mercer*, Journal on the London Math. Soc. **3** (1928), 130–134.

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