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A FEW REMARKS ON AUTOMORPHIC FUNCTIONS

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Let E and S be nonempty and suppose that g is a bijection of the set E. Following Ghermănescu [**1**] and Kuczma [**2**], we shall call solutions of the equation in $f : E \to S$

$$f(g(x)) = f(x) \quad (g,\, E,\, S \text{ given}) \tag{1}$$

automorphic functions.

The equation (1) is possible, since it is satisfied by the function f, defined by $f(x) = x_0$, where $x_0 \in S$ is fixed. The problem is to determine the general solution of (1).

If g generatés a finite group, i e if there exists a positive integer n such that $g^n(x) \equiv x$ for all $x \in E$, the general solution of (1) is easily constructed (in the case when E is a commutative ting in which every equation $nx = a$ has unique solution). namely, it is well-known that it is given by

$$f(x) = \sum_{v=0}^{n-1} \Pi(g^v(x)), \tag{2}$$

where $\Pi : E \to S$ is an arbitrary function.

However, if g generates an infinite group, the general solution of (1) cannot be obtained in closed form, though a number of expressions for the solution of (1) are known. We give three examples.

(i) According to [**1**] and [**2**], Popovici [**3**] gave the following formula for the general *number-valued* solution of (1):

$$f(x) = \sum_{v=-\infty}^{+\infty} \Pi(g^v(x)), \tag{3}$$

where $\Pi : E \to S$ is arbitrary function, such that the series in (3) converges.

Remark. Formula (3) is clearly a direct extension of the formula (2), and it is readily verified that (3) is a solution of (1). However, in the case when $g^n(x) \equiv x$, it is easly shown that any solution of (1) must have the form (2), whivh is not so for the solution (3).Besides, the reference [**3**], given in [**1**] and [**2**], is not correct—such a paper does not exist.

(ii) Kuczma [**4**] obtaind, under certain conditions, the general solution of the equation

$$f(g(x)) = G(x, f(x)), \tag{4}$$

and in the special case, for $G(s,t) = t$, we can get the general solution of (1). We shall not reproduce Luczma's result here, since it cannot be formulated without a substainal number of definitions and concepts; we only mentional that his proof is dependent on the Axiom of choice.

(iii) Prešić [**5**] writes the general solution of (1) in the form

$$f(x) = M \bigcup_{v=-\infty}^{+\infty} \{\Pi(g^v(x))\}, \tag{5}$$

where $\Pi : E \to S$ is arbitrary function, and $M : \mathbf{P}\, S \to S$ is a mapping such that $M\{x\} = x$. The existence of such a mapping follows from the Axiom of choice.

In this note we shall give an expression for the general solution of (1) which we have not found in literature.

Denote by $s(S)$ the set of all sequences of elements of S. We define the mapping $F : s(S) \to S$ in the following way:

If $(x_1, x_2, \dots) \in s(S)$, and if $x_0 \in S$ is fixed, then:

$F(x_1, x_2, \dots) = x_1$ if $(x_1, x_2, \dots)$ is a constant sequence, i.e. if $x_1 = x_2 = \dots$

$F(x_1, x_2, \dots) = x_0$ if $(x_1, x_2, \dots)$ is not a constant sequence.

The general solution of (1) is given by

$$f(x) = F(\Pi(x),\ \Pi(g(x)),\ \Pi(g^{-1}(x)),\ \Pi(g^2(x)),\ \Pi(g^{-2}(x)), \dots), \tag{6}$$

where $\Pi : E \to S$ is, as defore, an arbitrary function.

Indeed, the function f defined by (6) satisfies the equation (1). Moreover, if f_0 is a solution of (1), then choosing $\Pi = f_0$, from (6) follows $f(x) = f_0(x)$, which means that the solution is general.

Remark. The procedure described here, i.e. the use of the function F, has certain adwantages over known methods. First, it can be generalised to possible functional equations of the form

$$f(g(x)) = H(f(x)), \tag{7}$$

where H is invertible, which is not the case with Prešić's approach. Secondly, though the equations (7) is not as general as the equation (4), its solution can be obtained without the use of the Axiom of choice.

Generally speaking, to solve an equation means to find an equation equivalent to it which is "simple enough" so that we can call it the solution of the proposed equation. The solutions (3), (5), (6), or Kuczma's solution, though they give an explicit expression for $f(x)$, are not so simple that they can be *used in practice* as solutions of (1). Indeed, when we deal with a function f which is such that

$$f(x+2\pi)=f(x) \quad (E=S=\mathbf{R}), \tag{8}$$

we do not say that $f(x)=\sum_{v=-\infty}^{+\infty}\Pi(x+2v\pi)$, nor $f(x)=M\bigcup_{v=-\infty}^{+\infty}\{\Pi(x+v\pi)\}$, nor $f(x)=F(\Pi(x),\,\Pi(x+2\pi),\ \Pi(x-2\pi),\ \Pi(x+4\pi),\ \Pi(x-4\pi),\dots)$, where Π, M, F have the same meaining as above, but rather that f is "periodic with period 2π", which amounts to nothing but an other way of reading out (8).

This indicates that the equation (1) is what might be called a fundamental, basic, or a defining equation.

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