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A TABLEAUX SYSTEM IN MODAL LOGIC

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Tableaux proof are frequently used in discussing modal calculi. A method to be described here and applied to the Kripke modal system G (see more about G in [**1**]) is a modification of analytic tableaux develipoed by Smullyan in [**2**]. Its main feature, is the use of prefixed formulae[1] i.e. triples of the form (s, T, φ), (s, F, φ), (s^*, T, φ) or (s^*, F, φ) with s a (possibly empty) finite sequence of natural numbers and φ a formulae where formulae of G (denoted by $\varphi, \psi, \theta, \dots$) are defined as usual starting with a countable set of propositional letters and using, say, aconstant $\perp$, a unary propositional conective $\Box$ and a binary conective $\to$. The intended meaning of $s\,T(F)\varphi^2$ $s^*T(F)\varphi)$ is: φ is true (false) in the possible world $s\,(\varphi$ is true in (false) in all possible worlds related to s).

Let us now define a *tableau for* φ as any sequence $T_0, T_1, \dots$ of sets of sets of prefixed formulae such that $T_0 = \{\{\varnothing F\varphi\}\}$ and for all $m \in \omega$, all formulae ψ, θ and all $B \in T_m$ either

1° $sF\psi \to \theta \in B$ and $T_{m+1} = (T_m - \{B\}) \cup \{(B - \{sF\psi \to \theta\}) \cup \{sT\psi,\, sF\theta\}\}$ or

2° $s^*F\psi \to \theta \in B$ and T_{m+1} as above with s replaced by s^* or

3° $sT\psi \to \theta \in B$ and $T_{m+1} = (T_m - \{B\}) \cup \{(B - \{sT\psi \to \psi\}) \cup P | P \in \{\{sF\psi\}, \{sT\theta\}\}\}$ or

4° $s^*T\psi \to \theta \in B$ and for some $s' \subsetneq s$ occuring in $T_m T_{m+1} = (T_m - \{B\}) \cup \{B \cup P | P \in \{\{s'T\theta\}\}\}$ or

5° $sT\Box\psi \in B$ and $T_{m+1} = T_m - \{B\}) \cup \{(B - \{sT\Box\psi\}) \cup \{s^*T\psi\}\}$ or

6° $s^*T\Box\psi \in B$ and $T_{m+1} = (T_m - \{B\}) \cup \{B \cup \{s^{'*}T\psi | s' \subset s$ occurs in $T_m\}\}$ or

[1]The referee pointed out that prefixed formulae orginate with Anderson and Belnap (see their "Pure calculuc of entailment", J. Symp. Logic, 27, 19–52)

[2]Brackets and commas in prefixed formulae will always be omitted and s identified with its range

7° $sF\Box\psi\|B\|$ and $T_{m+1}=(T_m-\{B\})\cup\{(B-\{sF\Box\psi\})\cup P|P\in\{\{s'F\psi, s^{'*}\psi\}|$ $s' \underset{\neq}{\subset} s$ and either s' occurs in T_m or it is the (lexicorgraphically) first sequence extending $s\}\}$.

A set P of prefixed formulae is *closed* iff $T\ \bot\in P$ or for some φ $\{sT\varphi, sF\varphi\}\subseteq P$ or for some s and $\varphi s^*F\Box\varphi\in P$ and $\{s'\supset s|s'$ occurs in $P\}\neq\varnothing$ or for some $s'\underset{not=}{s}$ *upsets*$\{s^*T\varphi,\ s'F\varphi\}\subseteq P$ or $\{s^*F\varphi,\ s'T\varphi\}\subseteq P$

A tableau $T_0, T_1, \ldots$ is closed iff for some $i\in\omega$ all elements of T_i are closed; otherwise it is *open*. Call a formulae φ a *theorem* iff there is a closed tableau for φ.

Let $W\neq\varnothing$ and let $<$ be a binary relation on W. Then $(W,<)$ is a *frame for* G iff $<$ is transitive and well-founded.

Given a frame $(W,<)$ define a *valuation* on $(W,<)$ as any mapping v : $W\times$For$^3\to\{0,1\}$ satisfying: for all $W, W'\in W$ and all $\varphi,\psi\in$For:

1° $v(w,\bot)=0$;

2° $v(w,\ \varphi\to\psi)=1$ iff $v(w,\varphi)=0$ or $v(w,\psi)=1$

3° $v(w,\Box\varphi)=1$ iff for all $w'<w$ $v(w',\varphi)=1$.

Then call φ a *tautology* iff for all frames $(W,<)$ for G, for all $w\in W$ and all valuations v on $(W,<)\,v(w,\varphi)=1$.

We can now prove that all theorems for φ and v a valuation such that for some $w\in W$ $v(w,\varphi)=0$. Then for any tableau for φ and for all s occuring in it define a mapping $s\mapsto w(s)\in W$ by:

$w(\varnothing)=$the least $w\in W$ such that $v(W,\varphi)=0$,

w(si)=the least $w<w(s)$ such that for all φ if si$T(F)\varphi$ occurs in the tableau then $v(w,\varphi)=1(0)$ and if si$^*T(F)\varphi$ occurs in the tableau then for all $w'<w$ $v(w',\varphi)=1(0)$.

Now call an $sT\varphi$ v-true iff $v(w(s),\varphi)=1$ (other cases are similar) and also a set of prefixed formulae i v-true iff all its elements are. So in our case T_0 has a v-true member and if T_m has such a member it is proved (by checking all the cases) that sp does T_{m+1}. That would not be posibble if any of the closure conditions held (by definition of a valuation and well-foundedness of $<$), so the tableau must be open.

The converse also holds of the above result, i.e. all tautlogies are theorems. To rpove it suppose there is an open tableau for φ and extend it to a (finite) maximal one (in a sence that no more rules can be applied). Choose an open element B of the tableau. Then there is obviously a sequence $\{\varnothing F\varphi\}=B_0, B_1,\ldots,B_k=B(B_i\in T_i)$ of open sets such that we get B_{i+1} form B_i by application of one of the reduction rule $1°-7°$. Define a (finite) frame for G by $(\{s|s$ occurs in some $B_i\}\ \underset{\neq}{\supset})$ and a valuation v on it by $v(s,p)=1$ iff for some i $sTp\in B_i(p$ a propositional letter). Using closure conditions one can prove by induction on the complexity of

[3] The set of all formulae

formulae that all B_i are v-true, hence $v(\varnothing, \varphi) = 0$ i.e. φ is not a tautology.Notice that the whole procedure of searching for a counterexamples is effective.

REFERENCES

[1] Boolos, G., *The unprovability of consistency*, Cambridge, 1979.

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