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ON SUBSPACES OF RIEMANN-OTSUKI SPACE

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In this paper we observe an n-dimensional point-space determined with a given (1.1) tensor $P^i_j(x)$, $\det\|P^i_j\| \neq 0$, Otsuki's connection with the usual relations between P^i_j, ${}'\Gamma^i_{jk}$, and ${}''\Gamma^i_{jk}$ and the symmetric metric tensor gij, $\det\|g_{ij}\| \neq 0$, as in T. Otsuki [**2**] and A. Moór [**1**], but with the proposition $\gamma_k = 0$, i.e.

$$\nabla_k g_{ij} = \gamma_k g_{ij} = 0. \tag{1}$$

This space we call Riemann-Otsuki space $(R - O_n)$. According to the above proposition it follows that

$$''\Gamma^i_{jk} ='' \Gamma^i_{kj} = \left\{{i \atop jk}\right\}$$

where $\left\{{i \atop jk}\right\}$ denote Cristoffel symbols.

Let an m dimensional subspace S_m be defined as usual with $x^i = x^i(u^1, \dots, u^m)(m < n)$. We shall determine, through some assumptions, the basic elements of subspace S_m analogous to the tensor P^i_j and analogous to the coefficients of connections ${}'\Gamma^i_{j\ k}$ and ${}''\Gamma^i_{jk}$ of $R - O_n$, so that this subspace be a Riemann-Otsuki space $(R - O_m)$ too. By assumption rang $\left\| \frac{\partial x^i}{\partial u^\alpha} \right\| = m$[1]

Using the notation

$$\xi^i_\alpha : \frac{\partial x^i}{\partial u^\alpha} \tag{2}$$

we get the metric tensor $G_{\alpha\beta}$ of S_m by the requirement that *it is the projection of* g_{ij} *on* S_m. Hence

$$G_{\alpha\beta} = g_{ab}\xi^a_\alpha\xi^b_\beta; \quad G_{\alpha\beta}G^{\alpha\gamma} = \delta^\gamma_\beta; \quad G^{\alpha\gamma} = g^{ab}\xi^\alpha_a\xi^\gamma_b \tag{3}$$

where in the usual way we define

$$\xi^\alpha_i := g_{ij}G^{\alpha\beta}\xi^j_\beta \tag{4}$$

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[1]In this paper Latin indices run from 1 to n, Greek indices $\alpha, \dots, \varkappa$ run from 1 to m, but $\lambda, \mu, \dots$ run from $m+1$ to n.

and suppose that $\det \|G_{\alpha\beta}\| \neq 0$.

The projection on the basic tensor P^i_j of $R - O_n$ we denote by

$$P^\alpha_\beta : P^i_j \xi^\alpha_i \xi^j_\beta \tag{5}$$

and suppose that $\det \|P^\alpha_\beta\| \neq 0$, so that there are tensors $\overset{*}{Q}{}^\alpha_\beta$ satisfying

$$P^\alpha_\beta Q^\beta_\gamma = \delta^\alpha_\gamma. \tag{6}$$

The choice of the tensor P^α_β follows from our requirement that the S_m be an $R - O_m$ space. In the following *the tensor* P^α_β *of* S_m *will always be the projection of the tensor* P^i_j *of the basic space.*

If

$$T^i_j = T^\alpha_\beta \xi^i_\alpha \xi^\beta_j,$$

then the tensor T^i_j is a tensor of S_m. Now we can define the Otsuki covariant differential of the tensor T^α_β of S_m by

$$\begin{aligned} \overset{*}{D}T^{\alpha}\cdot &= P^\alpha_\gamma P^\delta_\beta T^\gamma_{\delta|\chi}\, du^\varkappa \\ &= P^\alpha_\gamma P^\delta_\beta (\partial_\varkappa T^\gamma_\delta +' \overset{*}{\Gamma}{}^\gamma_{\xi\varkappa} T^\xi_\delta -'' \overset{*}{\Gamma}{}^\xi_{\delta\gamma} T^\gamma_\xi)\, du^\varkappa = \overset{*}{\nabla} T^\alpha_b eta\, du^\varkappa. \end{aligned} \tag{7}$$

Here the coefficients ${}'\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ and ${}''\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ will be defined by the following suppositions:

(i) For the metric tensor $G_{\alpha\beta}$ we have the relation

$$\overset{*}{\nabla}_\varkappa G_{\alpha\beta} = P^\gamma_\alpha P^\delta_\beta (\partial_\varkappa G_{\gamma\delta} -'' \overset{*}{\Gamma}{}^\xi_{\gamma\ \varkappa} G_{\epsilon\delta} - \overset{*}{\Gamma}{}^\epsilon_{\delta\varkappa} G_{\gamma\xi}) = 0. \tag{8}$$

(ii) Between the tensor P^α_β and the coefficients of connections, ${}'\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ and ${}''\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$, we have the relation

$$P^{\beta}_\delta {}'' \overset{*}{\Gamma}{}_\beta{}^{alpha}{}_\gamma - P^{\alpha}_\xi {}' \overset{*}{\Gamma}{}_\delta{}^\varphi{}_\gamma + \partial_\gamma P^\alpha_\delta = 0. \tag{9}$$

Now we determine the coefficients of connection ${}''\overset{*}{\Gamma}{}_\beta{}^{alpha}{}_\gamma$. Using relation (8), according to the supposition $\det \|P^\alpha_\beta\| \neq 0$ and relations (6), (3) and (2) we get

$${}''\overset{*}{\Gamma}{}_\gamma{}^\xi{}_\varkappa G_{\xi\delta} +'' \overset{*}{\Gamma}{}_\delta{}^\xi{}_\varkappa G_{\gamma\xi} = (\partial_k g_{ij}) \xi^k_\varkappa \xi^i_\delta \xi^j_\gamma + g_{ij} (\xi_{delta\,\varkappa}^{\ i} \xi^j_\gamma + \xi^i_\delta \xi_{\gamma\,\varkappa}^{\ j}) \tag{10}$$

where

$$\xi_{\gamma\varkappa}^{\ i} = \frac{\partial}{\partial u^\varkappa} \xi^i_\gamma = \xi_{\varkappa\,\gamma}^{\ i}. \tag{11}$$

Now we construct the connection between the coefficients ${}''\Gamma_j{}^i{}_k$ and ${}''\overset{*}{\Gamma}{}_\beta{}^{alpha}{}_\gamma$. According to (1) we have

$$\partial_k g_{ij} =''\, \Gamma_i{}^s{}_k g_{sj} +''\, \Gamma_j{}^s{}_k g_{is}.$$

Substituting this in (10) we get

$$
\begin{aligned}
(12)\qquad {}''\overset{*}{\Gamma}{}_{\gamma}{}^{\xi}{}_{\varkappa}G_{\xi\delta} + {}''\overset{*}{\Gamma}{}_{\delta}{}^{\xi}{}_{\varkappa}G_{\gamma\xi} = {}''\Gamma_{i}{}^{s}{}_{k}g_{sj}\xi^{k}_{\varkappa}\xi^{i}_{\delta}\xi^{j}_{\gamma} + {}''\Gamma_{j}{}^{s}{}_{k}g_{si}\xi^{k}_{\varkappa}\xi^{i}_{\delta}\xi^{j}_{\gamma} +& \\
+ g_{ij}(\xi_{\delta}{}^{i}{}_{\varkappa}\xi^{j}_{\gamma} + \xi^{i}_{\delta}\xi_{\gamma}{}^{j}{}_{\varkappa}).&
\end{aligned}
$$

In the following we use the notation

$$
(13)\qquad {}''\Gamma_{\beta}{}^{\alpha}{}_{\gamma} = {}''\Gamma_{s}{}^{i}{}_{k}\xi^{\alpha}_{i}\xi^{s}_{\beta}\xi^{k}_{\gamma}.
$$

Applying (13) on (12) and making the contraction with metric tensor we get the relation symmetric in γ, *delta*:

$$
(14)\qquad {}''\overset{*}{\Gamma}_{\gamma\delta\varkappa} + {}''\overset{*}{\Gamma}_{\delta\gamma\varkappa} = {}''\Gamma_{\gamma\delta\varkappa} + {}''\Gamma_{\delta\gamma\varkappa} + g_{ij}(\xi_{\delta}{}^{i}{}_{\varkappa}\xi^{j}_{\gamma} + \xi^{i}_{\delta}\xi_{\gamma}{}^{j}{}_{\varkappa}).
$$

In the following we suppose that

$$
(15)\qquad {}''\overset{*}{\Gamma}_{\gamma\delta\varkappa} = {}''\overset{*}{\Gamma}_{\varkappa\delta\gamma}.
$$

Now we use the cyclic permutation of indices γ, *delta*, $\varkappa$ in (14) and subtract one of the equations from the sum of the other two. At the same time we use the symmetry of ${}''\Gamma_{j}{}^{i}{}_{k}$ and $\xi_{\alpha\beta}{}^{i}$ in the lower indices, and relation (15). So we get

$$
(16)\qquad {}''\overset{*}{\Gamma}_{\delta\varkappa\gamma} = {}''\Gamma_{\delta\varkappa\gamma} + \xi_{delta}{}^{j}{}_{\gamma}\xi^{i}_{\varkappa}g_{ij}.
$$

It is known that relation (2) dermines m tangent vectors of S_m. With the equations

$$
(17)\qquad \xi^{i}_{\alpha}N^{\mu}_{i} = 0;\ g_{ij}(x)N^{\mu}_{i}N^{\lambda}_{j} = \delta^{\mu\lambda} \qquad (\delta^{\mu\lambda}\ \text{ is the Kronecker symbol})
$$

we determine $n-m$ mutally ortogonal unit vectors which are orthogonal to S_m. In the case that $m=n-1$ we have only one vector of this kind, but if $m<n-1$, there are many possibilities for choosing them. We suppose that by the tangent vectors ξ^{i}_{α} the normal vectors N^{μ}_{i} are given too. Now we have the relation

$$
(18)\qquad \xi^{i}_{\alpha}\xi^{\alpha}_{j} + N^{i}_{\mu}N^{\mu}_{j} = \delta^{i}_{j}.
$$

Applying the contraction with $G^{\varkappa\xi}$ on (16), according to relations (13), (3), (18) and (17) we get

$$
(19)\qquad {}''\overset{*}{\Gamma}{}_{\delta}{}^{\varepsilon}{}_{\gamma} = {}''\Gamma_{\delta}{}^{\varepsilon}{}_{\gamma} + \xi_{\delta}{}^{i}{}_{\gamma}\xi^{\varepsilon}_{i} = {}''\Gamma_{\delta}{}^{\varepsilon}{}_{\gamma} - \xi^{i}_{\delta}\xi_{i}{}^{\varepsilon}{}_{\gamma}.
$$

Hence from the above considerations it follows that

THEOREM 1: *If relation* (15) *holds, then* (19) *is a necessary and sufficient conditions for* (8).

According to supposition (1) the coefficients ${}''\Gamma_{j}{}^{i}{}_{k}$ of $R-O_n$ are Christoffel symbols (see [**1**] (2.3)). From (8) it follows that the coefficients ${}''\overset{*}{\Gamma}{}_{\beta}{}^{\alpha}{}_{\gamma}$ defined by

(19) are, with respect to the metric tensor $G_{\alpha\beta}$, Christoffel symbols too and so we have

Corollary 1: *The relation* (19) *is the relation between Christoffel symbols of the second kind of $R-O_n$ and $R-O_m$, i.e.*

$$\{\delta\varepsilon\gamma\}_G = G_{\varepsilon\,\alpha}\left\{{\alpha \atop \delta\,\gamma}\right\} = \{\delta\varepsilon\gamma\}_g + \xi^{\;i}_{\delta\,\gamma}\xi^j_\varepsilon g_{ij}{}^2.$$

Proof: According to (6), from (8) it follows that

$$\partial_\varkappa G_{\delta\gamma} - {}''\overset{*}{\Gamma}{}^{\alpha}_{\delta\,\varkappa} G_{\alpha\gamma} - {}''\overset{*}{\Gamma}{}^{\;\alpha}_{\gamma\;\varkappa} G_{\delta\alpha} = 0. \tag{20}$$

Obviously it holds that ${}''\overset{*}{\Gamma}{}^{\;\varepsilon}_{\varkappa\,\gamma} = G^{\varepsilon\,\alpha}\{\varkappa\alpha\gamma\}_G$ is a solution of (8) resp (20). Now we prove the uniqueness of this solution. We suppose that

$${}''\overset{\sim}{\overset{*}{\Gamma}}{}^{\;\varepsilon}_{\varkappa\,\gamma} = G^{\varepsilon\alpha}\{\varkappa\alpha\gamma\}_G + \Lambda_{\varkappa\alpha\gamma}G^{\varepsilon\alpha}$$

is a solution too, where ${}''\overset{\sim}{\overset{*}{\Gamma}}{}^{\;\varepsilon}_{\varkappa\;\gamma}$ is symetric in $\varkappa$, γ, and $\Lambda_{\varkappa\alpha\gamma}$ is a tensor which according to (20) must be skew-symmetric in $\varkappa$, α and by the supposition symmetric in $\varkappa$, γ. Now we can write

$$\Lambda_{\varkappa\delta\gamma} = \Lambda_{\gamma\delta\varkappa} = -\Lambda_{\delta\gamma\varkappa} = -\Lambda_{\varkappa\gamma\delta} = \Lambda_{\gamma\varkappa\delta} = \Lambda_{\delta\varkappa\gamma} = -\Lambda_{\varkappa\delta\gamma}.$$

From this it follows that

$$\Lambda_{\varkappa\delta\gamma} = 0,$$

i.e.

$${}''\overset{\sim}{\overset{*}{\Gamma}}{}^{\;\varepsilon}_{\varkappa\,\gamma} = G^{\varepsilon\alpha}\{\varkappa\alpha\gamma\}_G = \left\{{\varepsilon \atop \varkappa\;\gamma}\right\}.$$

So we have proved that $\left\{{\varepsilon \atop \varkappa\,\gamma}\right\}_G$ is the unique solution of (20) resp (8). Since (19) is the solution of (8) too the assertion of Corollary 1 follows.

Using the results above we can determine the relation between DT_i and $\overset{*}{D}T_\alpha$, if T_i is a tensor of S_m. We have

Theorem 2: *In case T_i is a vector of S_m, i.e.*

$$T_i := \xi^\alpha_i T_\alpha, \tag{21}$$

we have

$$DT_\alpha = \xi^i_\alpha DT_i. \tag{22}$$

Proof: Applying the Otsuki invariant differential analogous to (7) on T_i and using (21), (5) and (7) we get

$$\begin{aligned}\xi^i_\alpha DT_i &= \xi^i_\alpha P^s_i(\partial_k T_s - {}''\,\Gamma^{\;j}_{s\;k}T_j)\xi^k_\beta\, du^\beta = \\ &= \xi^i_\alpha P^s_i \xi^\gamma_s(\partial_\beta T_\gamma - {}''\,\overset{*}{\Gamma}{}^{\;\delta}_{\gamma\,\beta}T_\delta)\, du^\beta = P^\gamma_\alpha T_{\gamma|\beta}\, du^\beta = \overset{*}{D}T_\alpha\end{aligned}$$

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[2] $\{\delta\varepsilon\gamma\}_g := \{ijk\}_g\ \xi^i_\delta\ \xi^j_\varepsilon\ \xi^k_\gamma$

where

$$''\overset{*}{\Gamma}{}_{\gamma\,\beta}^{\,\delta}\xi_s^\gamma =''\Gamma_{s\,\beta}^{\,delta} - \partial_\beta\xi_s^\delta \tag{23}$$

which is after one contraction with ξ_ε^s identical with (19).

Now we return to our (ii) property and determine the coefficients $'\overset{*}{\Gamma}{}_{\beta\gamma}^{\,\alpha}$ of the connection on m-dimensional subspace. It is known (se [**2**] (3.13)) that the relation between the tensor P_β^α and the coefficients $'\overset{*}{\Gamma}{}_{\beta\gamma}^{\,\alpha}$ and $''\overset{*}{\Gamma}{}_{\beta\gamma}^{\,alpha}$ is relation (9). From this relation using (6), (19), (5), (18), (17) and the fact that P_j^i, $'\Gamma_{j\,k}^{\,i}$ and $''\Gamma_{j\,k}^{\,i}$ are basic elements of $R-O_n$, i.e. satisfying a relation analogous to (9), we get:

$$'\overset{*}{\Gamma}{}_{\delta\gamma}^{\,\beta} = \overset{*}{Q}{}_\alpha^\beta\xi_i^\alpha(\xi_\delta^j\xi_\gamma^k P_a^{i\prime}\Gamma_{j\,k}^{\,a} - P_j^a\xi_\delta^j N_a^\mu N_\mu^{b\prime\prime}\Gamma_{b\,c}^{\,i}\xi_\gamma^c - P_j^a\xi_\delta^j N_a^\mu(\partial_\gamma N_\mu^i) + P_j^i\xi_{\delta\,\gamma}^{\,j}). \tag{24}$$

Now from the above considerations it follows that

Theorem 3: *If* (5), (7), (15), (19) *and* (24) *are satisfied, then the m-dimensional subspace of $R-O_n$ defined by $x^i = x^i(u^1,\dots,u^m)$ is an $R-O_m$.*

Using definition (7) one can prove that the relation analogous to (22) for the contravariant tensor

$$T6i = \xi_\alpha^i T^\alpha \tag{25}$$

of S_m generally does not hold. Now we shall construct the conditions by which this analogy holds. We can suppose that in place of relation (9) for tensor satisfying (25) we have

$$\overset{*}{D}T_\alpha = \xi_i^\alpha DT^i. \tag{9'}$$

Using Otsuki's covariant differential, relations (25) and (2) we get

$$\xi_i^\alpha DT^i = \xi_i^\alpha P_r^i((\partial_\varkappa T^\beta)\xi_\beta^r + (\partial_\varkappa\xi_\beta^r)T^\beta +'\Gamma_{s\,k}^{\,r}T^\beta\xi_\beta^s\xi_\varkappa^k)\,du^\varkappa$$

where

$$\xi_\beta^{r\prime}\overset{*}{\Gamma}{}_{\delta\,\varkappa}^{\,\beta} =' \Gamma_{s\,k}^{\,r}\xi_\varkappa^k\xi_\delta^s + \partial_\varkappa\xi_\delta^r.$$

One contraction by ξ_r^ε gives us

$$'\overset{*}{\Gamma}{}_{\delta\,\varkappa}^{\,\beta} =' \Gamma_{s\,k}^{\,r}\xi_\varkappa^k\xi_\delta^s\xi_r^\beta + \xi_r^\beta\xi_{\delta\,\varkappa}^{\,r} =' \Gamma_{\delta\,\varkappa}^{\,\beta} + \xi_r^\beta\xi_{\delta\,\varkappa}^{\,r}. \tag{26}$$

Now we have the question whether these coefficients with P_β^α and $''\overset{*}{\Gamma}{}_{\beta\gamma}^{\,\alpha}$ satisfy property (9) of Otsuki space. The answer is obviously no, but it is possible to find some special cases in them from (5), (7), (15), (19) and (26) follows that the m-dimensional subspace of $R-O_n$ is $R-O_m$. Substituting P_β^α, $''\overset{*}{\Gamma}{}_{\beta\gamma}^{\,\alpha}$ and $'\overset{*}{\Gamma}{}_{\beta\gamma}^{\,\alpha}$ from (5), (19) and (26) in the term on the left side of relation (9) we get

$$\begin{aligned} P_\delta^{\beta\prime\prime}\overset{*}{\Gamma}{}_{\beta\gamma}^{\,\alpha} - P_\varepsilon^{\alpha\prime}\overset{*}{\Gamma}{}_{\delta\gamma}^{\,\varepsilon} + \partial_\gamma P_\delta^\alpha = \xi_i^\alpha N_\mu^r\xi_\delta^b\xi_\gamma^c N_a^\mu(P_a^{i\prime}\Gamma_{b\,c}^{\,a} - \\ - P_b^{a\prime\prime}\Gamma_{a\,c}^{\,i}) + N_\mu^r N_a^\mu(P_r^s\xi_s^\alpha\xi_{\delta\,gamma}^{\,a} + P_b^a\xi_\delta^b\xi_{r\,\varkappa}^{\,\alpha}). \end{aligned} \tag{27}$$

Relation (9) will be satisfied if this expression vanishes.

I. If we suppose that $P^i_j = \varrho\delta^i_j$, $\varrho = \varrho(x)$, then (27) vanishes and relation (9) is satisfied, but it is known that then $'\Gamma^{\,i}_{j\,k} ='' \Gamma^{\,i}_{j\,k}$ and Otsuki space reduces to almost simple affin space.

bf II. Now we suppose that $m = n - 1$ and the normal vectors N_i are eigenvectors, i.e. $P^i_j N_i = \tau N_j$. In this case

$$P^i_r N^r = p^i_r (g^{rj} N_j) = P^j_r g^{ri} N_j = \tau N_r g^{ri} = \tau N^i$$

according to the supposition $P^i_j g_{ia} = P_{ja} = P_{aj}$. Substituting in (27) we get that relation (9) is satisfied.

III. In general for subspaces characterized with $P^i_j = P_] beta^\alpha \xi^i_\alpha \xi^\beta_j$ relation (9) is satisfied.

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