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FIXED-POINT MAPPINGS ON COMPACT METRIC SPACES

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Let (M, d) be a metric space and T a selfmapping of M into itself. If

$$d(Tx, Ty) < d(x, y) \tag{1}$$

holds for every x, y in M with $x \neq y$, then T is called a contractive mapping. On complete metric spaces contractive mappings may be without fixed-point. However, if M is compact, then every contractive selfmapping on M has a unique fixed point.

D. Bailey in [**1**] has proved that if M is compact and T is continuous and such that for every x, $y \in M$ with $x \neq y$ there exists a positive integer $n(x, y)$ such that

$$d(T^{n(x,y)}x,\ T^{n(x,y)}y) < d(x, y), \tag{2}$$

then T has a unique fixed point in M.

In the following we will extend the result of Bailey to mappings which satisfy a contractive condition which is weaker then (2).

We now prove the following theorem.

THEOREM 1. *Let T be a continuous mapping on the compact metric space M into itself satisfying the inequality*

$$d(T^n x,\ T^n y) < \max\Big\{d(x,\ y),\ d(x,\ Tx),\ d(y,\ Ty),\ \frac{1}{2}\Big[d(x,\ Ty) + d(y,\ Tx)\Big]\Big\} \tag{3}$$

for all x, y in M with $x \neq y$ where $n = n(x, y)$ is a positive integer. Then T has a unique fixed point in M.

Proof. Define on M a real-valued function F by $F(x) = d(x, Tx)$. Since T is continous, it follows that F is continous too. Therefore, F on M attains its maximum and minimum. Let u in M be such that

$$F(u) = \min\{F(x) : x \in M\}. \tag{4}$$

We will show that u is a fixed-point of T. If we assume that $F(u) = d(u, Tu) > 0$, then by (3), for $n = n(u, Tu)$, we have

$$\begin{aligned} F(T^n u) &= d(T^n u,\ TT^n u) = d(T^n u,\ T^n Tu) \\ &< \max\Big\{d(u, Tu),\ d(u, Tu),\ d(Tu,\ TTu),\ \frac{1}{2}\Big[d(u,\ T^2u) + 0\Big]\Big\} \\ &\le \max\Big\{F(u),\ F(Tu),\ \frac{1}{2}\Big[F(u) + F(Tu)\Big]\Big\}. \end{aligned}$$

Since by (4) $\max\big\{F(u), F(Tu), \frac{1}{2}\Big[F(u) + F(Tu)\Big]\big\} = F(u)$, we have $F(T^n u) < F(u)$, which is a contradiction with (4). Therefore, u is a fixed-point of T. The uniqueness of u follows easily from (3). This completes the proof of the theorem.

Theorem 1 holds if some of the conditions are relaxed. So we have

THEOREM 2. *Let T be an orbitally continous mapping on the compact metric space M into itself satisfying* (3). *Then T has a unique fixed-point in M.*

The following example shows that the continuity conditions of T in the theorems 1. and 2. cannot be removed.

Example. If M is the clossed interval $[0, 1]$ and $T : M \to M$ is defined by $T(x) = \frac{x}{2}$ if $x \neq 0$ and $T(0) = 1$, then T satisfies (3) with $n(x, y) = 2$, as $d(T^2 0, T^2 x) = \frac{1}{2} - \frac{x}{4} < \frac{1}{2} \cdot 1 = \frac{1}{2} d\ (0, T0)$ and $d(T^2 y,\ T^2 x)\frac{1}{4} d(y, x)$ for $y \neq 0$. However, T has not fixed points.

Now we will show that the condition (3) may be much more weakened.

THEOREM 3. *Let T be a continous mapping on the compact metric space M into itself satisfying the inequality*

$$\begin{aligned} d(T^n x,\ T^n y) < \max_{0 \le p,q,r,s,t \le n} \Big\{&d(T^p x,\ T^p y),\ d(T^q x,\ T^{q+1} x), \\ &d(T^r y,\ T^{r+1} y),\ \frac{1}{2}[d(T^s x,\ T^{s+1} y) + d(T^t y,\ T^{t+1} x)]\Big\} \end{aligned} \tag{5}$$

for some positive integer $n = n(x, y)$ and x, y in M for which the right-hand side of inequality is positive. Then T has a unique fixed-point in M.

Proof. We may assume that the right-hand side of inequality (5) is positive for each x, y in M. For if it is not positive, then $x = y = Tx$, which means that T has a fixed-point. Let the mapping F and the point u be defined as in the proof of the theorem 1. and let $n = n(u, Tu)$. Then by (5) it follows

$$\begin{aligned} F(T^n u) = d(T^n u,\ T^n Tu) < \max_{0 \le p,q,r,s,t \le n} \Big\{&d(T^p u,\ TT^p u),\ d(T^q u,\ TT^q u), \\ &d(T^{r+1} u,\ TT^{r+1} u),\ \frac{1}{2}[d(T^s u,\ T^2 T^s u) + 0]\Big\}. \end{aligned}$$

Using the triangle inequality and (4) we obtain $F(T^n) < F(u)$, which is a contradiction with (4). Therefore, the right-hand side of (5) is zero for $x = u$ and $y = Tu$. Hence $Tu = u$. The uniqueness of the fixed-point follows easily. This completes the proof of the theorem.

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