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A NEW FIXED-POINT THEOREM FOR CONTRACTIVE MAPPING

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Let (M, d) be a metric space and let T be a mapping of M into itself. A mapping T is called contractive if it satisfies

$$d(Tx, Ty) < d(x, y)$$

for all x, y in M with $x \neq y$. A Meir and E. Keeler [**1**] have introduced a very weak contractive condition which guarantees the existence of a fixed point in complete metric spaces. Their conctractive condition is the following:

$$\text{(1)} \qquad \begin{gathered} \text{Given } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ \varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(Tx, Ty) < \varepsilon. \end{gathered}$$

We now prove the following fixed point theorem.

THEOREM. *Let M be a complete metric space and let T be a mapping of M into itself satisfying the condition*

$$\text{(2)} \qquad \begin{gathered} \textit{Given } \varepsilon > 0 \textit{ there exists } \delta > 0 \textit{ such that} \\ \varepsilon < d(x, y) < \varepsilon + \delta \textit{ implies } d(Tx, Ty) \leq \varepsilon. \end{gathered}$$

Then T has a unique fixed-point u in M and $\lim_{n\to\infty} T^n x = u$ for each x i M.

Proof. Let x in M be arbitrary and consider the sequence

$$x_0 = x, \;\; x_1 = Tx_0, \ldots, x_n = T^n x, \ldots.$$

Since (2) implies that T is conctractive, it follows that the real sequence $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ is nonincreasing.

If $d(x_n, x_{n+1}) = d(x_n, Tx_n) = 0$ for some n, then x_n is a fixed-point of T and the proof is finished. Assume now that $d(x_n, x_{n+1}) > 0$ for each $n = 0, 1, 2, \ldots$. Then the real sequence $\{d(x_n, x_{n+1})\}_{n=0}^{\infty}$ is strictly decreasing and therefore has a limit $\varepsilon_0 \geq 0$. By monotonicity we have

$$\text{(3)} \qquad d(x_n, x_{n+1}) > \varepsilon_0 \text{ for } n = 0, 1, 2, \ldots$$

Assume that $\varepsilon_0 > 0$. Then $\delta_0 = \delta(\varepsilon_0) > 0$ and there is some k such that

$$\varepsilon_0 < d(x_k, x_{k+1}) < \varepsilon_0 + \delta_0.$$

By (2) we have $d(Tx_k, Tx_{k+1}) = d(x_{k+1}, x_{k+2}) \leq \varepsilon_0$, which is a contradiction with (3). Therefore, $\varepsilon_0 = 0$ and

$$\lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0. \tag{4}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary and let $\delta = \delta(\varepsilon)$ be choosen such that $\delta \leq \varepsilon$. By (4) there exists some K such that

$$d(x_{n-1},\ x_n) < \delta \ \text{ for each } \ n > K. \tag{5}$$

Fix $n > K$. It suffices to show that

$$d(x_n, x_{n+p}) \leq \varepsilon \ \text{ for } \ p = 1, 3, \ldots . \tag{6}$$

As for $p = 1$ (6) follows from (2) and (5) (since $\delta \leq \varepsilon$) we may proceed by induction on p. Assume that (6) holds for some fixed p. Then by (5) and (6) we have

$$d(x_{n-1}, x_{n+p}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+p}) < \delta + \varepsilon.$$

Hence by (2)

$$d(x_n, x_{n+p+1}) = d(Tx_{n+1}, Tx_{n+p}) \leq \varepsilon \tag{7}$$

in the case $d(x_{n+1},\ x_{n+p}) > \varepsilon$. In the case $d(x_{n+1},\ x_{n+p}) \leq \varepsilon$ (7) holds by contractivity of T. Thus we conclude by induction that (6) is valid for any $n > K$ and for each $p = 1, 2, \ldots$. Hence $\{x_n\}$ is a Cauchy sequence. As M is complete, there exists some u in M such that

$$u = \lim_{n\to\infty} x_n.$$

Then we have $Tu = u$ by continuity of T. The uniqueness of u follows easily. This completes the proof of the theorem.

The contractive conditions (1) and (2) are not equivalent. The following example shows that there are metric spaces which admit concrative mappings which satisfy our condition (2), but not Meir-Keeler's condition (1).

Example. Let

$$M = \left\{0, 1, 2, 3, 4 + \frac{1}{2}, \ldots, 3n,\ \ 3n + 1 + \frac{1}{n+1}, \ldots\right\}$$

be a subset of reals with the usual metric and let T on M be defined by

$$\begin{aligned} Tx &= 0, \quad \text{if } x = 0,\ 1,\ 3, \ldots, 3n, \ldots \\ Tx &= 1, \quad \text{if } x = 4 + \frac{1}{2},\ 7 + \frac{1}{3}, \ldots, 3n + 1 + \frac{1}{n+1}, \ldots . \end{aligned}$$

Then for each $\varepsilon > 0$ and any $\delta > 0$ the mapping T satisfies (2). However, T does not satisfy (1), as for $\varepsilon = 1$ there is no $\delta > 0$ such that (1) holds. For, if we assume that $\delta(1) > 0$, then we may choose a sufficiently large n such that

$$1 \leq d\left(3n,\ 3n + 1 + \frac{2}{n}\right) < 1 + \delta.$$

Then by (1) this implies $d\Big(T(3n),\ T(3n + 1 + \frac{1}{2})\Big) < 1$, which is incorrect.

REFERENCES

[1] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28** (1969), 326–329.

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