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ON A CLASS OF N-ARY QUASIGROUPS

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Let $(Q, \cdot)$ be a loop with the property:

(G) *For every loop $(H, *)$, if loops $(Q, \cdot)$ and $(H, *)$ are isotopic, they are isomorphic.*

A loop $(Q, \cdot)$ with the property (G) is called a G–loop [**1**].

By Albert's theorem, any group is a G-loop. A useful characterization of G-loops was given by V. D. Belousov [**1**] in terms of derived operations.

Let $(Q, \cdot)$ be a quasigroup, $a \in Q$. The operation ${}_a\cdot$ of the set Q defined by

$$x_a \cdot y = \varrho^{-1}(x \cdot \varrho_a y)$$

is called the left derived operation determined by a [**1**]. Analogously, the right derived operation determined by a is defined by

$$x \cdot_a y = \lambda_a^{-1}(\lambda_a x \cdot y)$$

V. D. Belousov obtained the following result.

A loop $(Q, \cdot)$ is a G–loop if and only if $(Q, \cdot)$ is isomorphic to all its left and all its right derived operations.

We establish here an analogous result for n-ary quasigroups. Previously, we give some definitions. The notation is standard in quasigroup theory [**2**]. If (Q, A) is an n-ary quasigroup, and $\bar{a} = a_1^{i-1} a_{i+1}^n \in Q^{n-1}$, then $L_i(\bar{a})x \overset{\mathrm{d}ef}{=} A(a_1^{i-1}, x, a_{i+1}^n)$.

Definition 1. Let (Q, A) be an n-ary quasigroup, and $\bar{a} \in Q^{n-1}$. The n-ary operation of the set Q defined by

$$A_{\bar{a}}^i(x_1^n) \overset{\mathrm{d}ef}{=} L_i^{-1}(\bar{a}) A([L_i(\bar{a})x_\alpha]_{\alpha=1}^{i-1},\ x_i,\ [L_i(\bar{a})x_\alpha]_{\alpha=i+1}^n)$$

is called the i-th derived operation of A determined by the sequence $\bar{a}, i = 1, \dots, n$.

If $n = 2$, we obtain the left and the right derived operations determined by the element a of Q:

$$A_a^1(x,y) = L_1^{-1}(a)A(x,\ L_1(a)\dot{y}),$$
$$A_a^2(x,y) = L_2^{-1}(a)A(L_2(a)x, y).$$

Since operations $A_{\bar{a}}^i$ are isotopic to the operation A, they all are quasigroups, too.

If (Q, A) ia an n-ary quasigroup, a sequence $\tilde{\mathfrak{e}} = e_1^{i-1}e_{i+1}^n \in Q^{n-1}$, such that

$$(\forall x \in Q) L_i(\tilde{\mathfrak{e}})x = x,$$

is called an *i–th identity sequence* of (Q, A).

LEMMA 1. *The n-ary quasigroup $(Q, A_{\bar{a}}^i)$ has an i-th identity sequence $(i = 1, \dots, n)$.*

Proof. If $e_\alpha \stackrel{\text{def}}{=} L_i^{-1}(\bar{a})a_\alpha,\ \alpha \in \{1, \dots, n\},\ \alpha \neq 1$, we have

$$A_{\bar{a}}^i(e_1^{i-1}, x, e_{i+1}^n) = L_i^{-1}(\bar{a})A(a_1^{i-1}, x, a_{i+1}^n) = L_i^{-1}(\bar{a})L_i(\bar{a})x = x.$$

The next lemma establishes a connection between derived operations and pseudo-automorphisms of an n-ary quasigroup [**3**].

LEMMA 2. *A sequence $\bar{a} \in Q^{n-1}$ is a companion of some i-th pseudo-automorphism φ of an n-ary quasigroup (Q, A), if and only if and only if the quasigroups (Q, A) and $(Q, A_{\bar{a}}^i)$ are isomorphic.*

Proof. φ is an i-th pseudo-automorphism with a companion $\bar{a}$.

$$\begin{aligned}
&\leftrightarrow L_i(\bar{a})\varphi A(x_1^n) = A([L_i(\bar{a})\varphi x_\alpha]_{\alpha=1}^{i-1},\ \varphi x_i,\ [L_i(\bar{a})\varphi x_\alpha]_{\alpha=i+1}^n)\\
&\leftrightarrow \varphi A(x_1^n) = A_{\bar{a}}^i(\varphi x_1, \dots, \varphi x_n)\\
&\leftrightarrow (Q, A) \cong (Q, A_{\bar{a}}^i).
\end{aligned}$$

Definition 2. An n-ary quasigroup (Q, A) is a generalized n-ary loop if for every $i = 1, \dots, n$ there exists an i-th identity sequence $\tilde{\mathfrak{e}}_i = [e_{i\alpha}]_{\alpha=1}^{i-1}[e_{i\alpha}]_{\alpha=i+1}^n, e_{ij} \in Q$.

Clearly, every n-ary loop [**2**] with an identity e, is a generalized n-ary loop with $\tilde{\mathfrak{e}}_i = \tilde{\mathfrak{e}} = \overset{n-1}{e}$. In the binary case, every generalized loop is a loop.

LEMMA 3. *If (Q, A) is an n-ary loop, then every derived quasigroup $(Q, A_{\bar{a}}^i)$ is a generalized n-ary loop.*

Proof. Let (Q, A) be an n-ary loop with an identity, e, and let $A_{\bar{a}}^i$ be a derived operation. By lemma 1, $A_{\bar{a}}^i$ has an i-th identity sequence. If $f = L_i^{-1}(\bar{a})e$, we have for every j, $1 \le j < i$,

$$A^i_{\bar a}(\overset{j-1}{f}, x, \overset{i-1-j}{f}, e, \overset{n-i}{f})$$
$$= L_i^{-1}(\bar a)A\,(\overbrace{L_i(\bar a)L_i^{-1}(\bar a)e}^{j-1},\ L_i(\bar a)x,\ \overbrace{L_i(\bar a)L_i^{-1}(\bar a)e}^{i-1-j},\ e,\ \overbrace{L_i(\bar a)L_i^{-1}(\bar a)e}^{n-i})$$
$$L_i^{-1}(\bar a)A(\overset{j-1}{e},\ L_i(\bar a)x, \overset{n-j}{e}) = L_i^{-1}(\bar a)L_i(\bar a)x = x,$$

hence, $A^i_{\bar a}$ has a j-th identity sequence, for $1 \le j < i$.

Similarly we prove that $A^i_{\bar a}$ has j-th identity sequences for $i < j \le n$, thus, $(Q, A^i_{\bar a})$ is a generalized n-ary loop.

Let (Q, A) be an n-ary quasigroup and let $\bar a_i = [a_{i\alpha}]^{i-1}_{\alpha=1}[a_{i\alpha}]^n_{\alpha=i+1}$, $i = 1, \dots, n$, $a_{i\alpha} \in Q$. We introduce the following operation of Q:

$$A_{\bar a_1 \cdots \bar a_n}(x_1^n) \overset{\mathrm{def}}{=} A(L_1^{-1}(\bar a_1)x_1, \dots, L_n^{-1}(\bar a_n)x_n),$$

which is a principal isotop of A. It is easy to verify that $(Q, A_{\bar a_1 \dots \bar a_n})$ is a generalized n-ary loop with i-th identity sequence

$$\tilde{\mathfrak{e}} = [L_\alpha(\bar a_\alpha)a_{i\alpha}]^{i-1}_{\alpha=1}[L_\alpha(\bar a_\alpha)a_{i\alpha}]^n_{\alpha=i+1}, \quad i = 1, \dots, n.$$

If $a_{i\alpha} = a_\alpha$, for $i = 1, \dots, n$, $\alpha \ne i$, then $(Q,\ A_{\bar a_1 \dots \bar a_n})$ is an n-ary loop with identity element $e = A(a_1^n)$. Indeed, then we have $L_\alpha(\bar a_\alpha)a_{i\alpha} = A(a_1^n)$, and if we put $e = A(a_1^n)$, then $\tilde{\mathfrak{e}}_i = \overset{n-1}{e}$, $i = 1, \dots, n$.

Lemma 4. *Let (Q, A) be an n-ary quasigroup. For every operation $A_{\bar a_1 \dots \bar a_n}$ there exist sequences $\bar b_1, \dots, \bar b_n$ of elements of Q such that $(Q, A_{\bar a_1 \cdots \bar a_n}$ and $(Q, (\cdots(A\frac{1}{b_1})\frac{2}{b_2}\cdots\frac{n}{b_n})$ are isomorphic.*

Proof. First, let $\bar b_1, \dots, \bar b_n$ be arbitrary elements of Q^{n-1}. By definition of derived operations, we have

$$(\cdots(A\frac{1}{b_1})\frac{2}{b}\cdots)\frac{n}{b}(x_1^n) = \dot L_n^{-1}\cdot\cdot\dot L_2^{-1}L_1A(\dot L_2\cdot\cdot\dot L_n x_1,\ L_1\dot L_3\cdot\cdot\dot L_n x_2, \dots, L_1\cdot\cdot\dot L_{n-1}x_n)$$

where

$$L_1x = L_1(\bar b_1)x = A(x, b_{12}, \dots, b_{1n}) <$$
$$\dot L_2 x = \dot L_2(\bar b_2)x = A\frac{1}{b_1}(b_{12},\ x, \dots, b_{2n}),$$
$$\dots$$
$$\dot L_n x = \dot L_n(\bar b_n)x = (\cdots(A\frac{1}{b_1})\frac{2}{b_2}\cdots)\frac{n-1}{b_n}(b_{n1}, \dots, b_{nn-1}, x).$$

By induction on k, we prove

$$L_1\dot L_2\cdots\dot L_k = \bar L_k\bar L_{k-1}\cdots L_1,\quad k = 2, 3, \dots, n, \tag{1}$$

where

$$\overline{L}_2 x = L_2(\overline{\tau_2 b_2})x = A(\tau_{21} b_{21},\ x,\ \dots, \tau_{2n} b_{2n}),$$

$$\dots\dots$$

$$\overline{L}_n x = L_n(\overline{\tau_n b_n})x = A(\tau_{n1} b_{n1},\ \dots, \tau_{nn-1} b_{nn-1}, x),$$

and τ_{ij} are certain bijections of Q, which depend only on $b_k,\ k < i$.

First we prove $L_1 \dot{L}_2 = \overline{L}_2 L_1$. By definition of $\dot{L}_2$, we have

$$\begin{aligned}\dot{L}_2 x &= L_1^{-1} A(b_{21}, L_1 x, \dots, L_1 b_{2n})\\ &= L_1^{-1} \overline{L}_2 L_1 x,\end{aligned}$$

hence, $L_1 \dot{L}_2 = \overline{L}_2 L_1$.

Next assume that $L_1 \dot{L}_2 \cdots \dot{L}_{k-1} = \overline{L}_{k-1} \cdots \overline{L}_2 L_1$. By definition of $\dot{L}_k$, it follows that

$$\dot{L}_k = \dot{L}_{k-1}^{-1} \cdots \dot{L}_2^{-1} \overline{L}_k L_1 \dot{L}_2 \cdots \dot{L}_{k-1},$$

which implies

$$L_1 \dot{L}_2 \cdots \dot{L}_k = \overline{L}_k \dot{L}_1 \cdots \dot{L}_{k-1}.$$

Hence, by the induction assumption, it follows (1).

Consequently, we obtain

$$(\cdots(A\frac{2}{b_1})\frac{2}{b}\cdots)\frac{n}{b}(x_1^n) = \delta^{-1} A(L_1^{-1}\delta x_1, \overline{L}_2^{-1}\delta x_2, \dots, \overline{L}_n^{-1}\delta x_n),$$

where $\delta = \overline{L}_n \cdots \overline{L}_2 L_1$. Thus, $(Q, (\cdots(A\frac{1}{b_1})\frac{2}{b_2}\cdots)\frac{n}{b})$ and $(Q, A_{\bar{a}_1 \cdots \bar{a}_n})$ are isomorphic, where

$$\begin{aligned}\bar{a}_1 &= \bar{b}_1\\ \bar{a}_2 &= \overline{\tau_2 b_2} = \tau_{21} b_{21} \tau_{23} b_{23} \cdots \tau_{2n} b_{2n}\\ &\dots\dots\\ \bar{a}_n &= \overline{\tau_n b_n} = \tau_{n1} b_{n1} \cdots \tau_{nn-1} b_{nn-1}\end{aligned}$$

and τ_{ij} are bijections of the set Q.

Now it follows that for arbitrary $\bar{a}_1, \dots \bar{a}_n$ there exist $\bar{b}_1, \dots, \bar{b}_n$, such that $\bar{b}_1 = \bar{a}_1$, $\bar{b}_2 = \overline{\tau_1^{-1} a_2}, \dots \bar{b}_n = \overline{\tau_n^{-1} a_n}$, and

$$\delta(\cdots(A\frac{1}{b_1})\frac{2}{b_n}\cdots)\frac{n}{b}(x_1^n) = A_{\bar{a}_1 \cdots \bar{a}_n}(\delta x_1, \dots, \delta x_n).$$

Lemma 5. *If a generalized n-ary loop (H, B) is isotopic to an n-ary quasigroup (Q, A), then there exist sequences $\bar{a}_1, \dots, \bar{a}_n$ of elements of Q such that (H, B) and $(Q, A_{\bar{a}_a \cdots \bar{a}_n})$ are isomorphic.*

Proof. Let $\alpha_{n+1} B(x_1^n) = A(\alpha_1 x_1, \cdots, \alpha_n x_n)$, and let $\tilde{e}_i,\ i = 1, \dots, n$, be identity sequences of (H, B). Then we have

$$\begin{aligned}\alpha_{n+1} x_i &= A(\alpha_1 e_{i1}, \dots, \alpha_{i-1} e_{ii-1}, \alpha_i x_i,\ \alpha_{i+1} e_{ii+1}, \dots, \alpha_n e_{in})\\ &= L_i(\overline{\alpha e_i})\alpha_i x_i\end{aligned}$$

Hence, $\alpha_i x_i = L_i^{-1}(\overline{\alpha e_i})\alpha_{n+1}x_i,\ i = 1,\dots,n$, and

$$\alpha_{n+1}B(x_1^n) = A(L_1^{-1}(\overline{\alpha e_a})\alpha_{n+1}x_1,\dots,L_n^{-1}(\overline{\alpha e_n})\alpha_{n+1}x_n).$$

Thus, (H,B) and $(Q,A_{\overline{\alpha e_1},\dots,\overline{\alpha e_n}})$ are isomorphic.

An n-ary loop (Q,A) with the property

(G_n) *For every n-ary loop (H,B), if (H,B) and (Q,A) are isotopic, then they are isomorphic*

is called an$n$-aty G-loop.

Similarly, an n-aty loop with the property

(G'_n) *For every generalized n-ary loop (H,B), if (H,B) and (Q,A) are isotopic, then they are isomorphic*

is called a G'-loop.

Clearly, the property G'_n implies the property G_n. As an immediate consequence of definition of a G'-loop, it follows that every generalized loop, isotopic to a G'-loop is a loop, too.

THEOREM. *If (Q,A) ia an n-ary loop, then the following statements are equivalent:*

(i) (Q,A) is a G'-loop

(ii) (Q,A) is a G'-loop, and every derived quasigroup of (Q,A) is a loop.

(iii) (Q,A) is isomorphic to every derived quasigroup of $(Q,A\frac{i}{a})$, $\bar{a} \in Q^{n-1},\ i = 1,\dots,n$.

Proof. (i) $\Rightarrow$ (ii). Since all derived quasigroups are generalized loops (lemma 3), isotopic to (Q,A) they are isomorphic to (Q,A). Hence they are loops.

(ii) $\Rightarrow$ (iii). Trivially.

(iii) $\Rightarrow$ (i) Let (Q,A) be isomorphic to every derived quasigroup $(Q,A\frac{1}{a})$, and let (H,B) be a generalized loop, isotopic to (Q,A). By lemma 5, (H,B) is isomorphic to a principal isotop $(Q,A_{\bar{a}_1\cdots\bar{a}_n})$ of the loop (Q,A). On the other hand, by lemma 4, $(Q,A_{\bar{a}_1\cdots\bar{a}_n})$ is isomorphic to $(Q,(\cdots(A\frac{1}{b_1})\frac{2}{b_2}\cdots)\frac{n}{b})$, for some $\bar{b}_1,\dots,\bar{b}_n \in Q^{n-1}$. By (iii), (Q,A) is isomorphic to $(Q,(\cdots(A\frac{1}{b_1})\frac{2}{b_2}\cdots)\frac{n}{b})$. Consequently, (H,B) is isomorphic to (Q,A). Hence, (Q,A) is a G'-loop.

Example 1. Let (Q,A) be an n-ary loop satisfying i-th Menger's laws for all $i = 1,\cdots,n$ **[2]**. By definition of i-th derived operation, such a loop coincides with all its derived operations. Hence, it is a G'-loop.

Example 2. Let (Q,A) be an n-ary group with identity element e. According to Hosszu-Gluskin's theorem **[2]** there is a binary group $(Q,\cdot)$ such that $A(x_1^n) = x_1,\cdot\ldots\cdot x_n$. A straightforward verification shows that (Q,A) is a G-loop: if $\alpha_{n+1}B(x_1^n) = \alpha_1 x_1 \cdot \ldots \cdot \alpha_n x_n$, then $\varphi B(x_1^n) = \varphi x_1 \cdot \ldots \cdot \varphi x_n$, where

$\varphi = \lambda_c \alpha_{n+1}$, $c = (\alpha_1 e, \dots, \alpha_n e)^{-1}$. Generally, (Q, A) is not a G'-loop. Indeed, let Q^+ be the set of all nonnegative rational numbers. If $(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3$, then (Q^+, A) is a ternary group, with identity element 1, but there exist derived quasigroups which are not isomorphic to (Q^+, A). For example, if $\bar{a} = 1, 2$, we have $L_1(\bar{a})x = x \cdot 2 \cdot 1 = 2x$, $L_1^{-1}(\bar{a})x = 2^{-1}x$, $A_{\bar{a}}^1(x, y, z)) = 2xyz$, and $(\forall x \in Q^+) 2xyy = x \Rightarrow y = \sqrt{2}/2$. Hence, $(Q^+, A_{\bar{a}}^1)$ is a ternary group without identity element, and it is not isomorphic to (Q^+, A).

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