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OSCILLATIONS OF n-TH ORDER RETARDER DIFFERENTIAL EQUATIONS[1]

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1. Introduction

The purpose of this paper is to finding the oscillatory criteria for the following equation

$$y^{(s)}(t)+(-1)^{n+1}\sum_{i=1}^{m} f_i(t,y(t),y(g_i(t)))=h(t). \tag{1}$$

Throughout this paper, we assume that the following conditions are satisfied:

(a) $f_i \in C[R^+ \times R^2, R]$, $i, 2, \dots, m$, *and for some index* $j, 1 \le j \le m$, $f_j(t,u,v_j)$ *is increasing in* u *and* v_j *for fixed large* t.

(b) $f_i,(t,u,v_i)$, *has the same sign as that of* u *and* v, *for* $i=1,2,\dots,m$.

(c) $g_i \in C[R^+, R]$ *and* $g_i(t) \le t$, $g_i(t)$ *is nondecreasing*, $\lim_{t\to\infty} g_i(t)=\infty$ *for* $i=1,\dots,m$, *and* $g_j(t)$ *is strictly increasing, index* j *associate with* $f_j(t,u,v_j)$ *in* (a).

(d) *there exists a function* $r(t)$ *such that* $r^{(n)}(t)=h(t)$, $r^{(i)}(t)\to 0$ *as* $t\to\infty$, $i=0,\dots,n-1$.

In what follows, we consider only such solutions which are defined for all large t. The oscillatory character is considered in the usual sense, i.e. a continuous real-valued function which is defined for all large t is called *oscillatory* if it has no last zero, and otherwise it is called *nonoscillatory*.

Similar dicussions to that given here have been obtained in [1 - 5] for the solutions of the following retarded differential equations of the particular forms

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$$y''(t) - \sum_{i=1}^{m} p_i(t)y(g_i(t)) = 0,$$

$$y''(t) - \sum_{i=1}^{m} f_i(t, y(t), y(g_i(t))) = 0,$$

$$y^{(n)}(t) + (-1)^{n+1}p(t)y(g(t)) = 0,$$

$$y^{(n)}(t) + (-1)^{n+1}p(t)h(y[g(t)]) = 0,$$

and

$$y^{(n)}(t) + (-1)^{n+1}p(t)y(g(t)) = f(t).$$

2. Main Results

THEOREM 1. *Let the conditions* (a)–(d) *hold. Asume that* $y(t)$ *is a bounded solution of* (1) *with* $|y(t)| \leq L$, *for large* t, *and* $L > 0$. *Let there exist a nonempty set of indices* $K = \{c_1, \ldots, c_M\}$, $1 \leq c_1 \leq c_2 < \cdots < c_M \leq m$ *and functions* ${G_L}^i \in C[R^+, R^+]$, $i \in K$ *such that for* $v_i \neq 0$, $i \in K$ *and large* t

$$\text{(2)} \qquad {G_L}^i(t) \leq {v_i}^{-1} f_i(t, u, v_i).$$

Suppose

$$\text{(3)} \qquad \liminf_{t\to\infty} \int_{g^*(t)}^{t} \sum_{i\in K} [g_i(t) - g_i(s)]^{n-1} {G_L}^i(s)ds > 1$$

where $g^*(t) = \max_{i\in K}\{g_i(t)\}$. *If*

$$\varphi(t) = \sum_{i\in k} \int_{g^*(t)}^{t} r(g_i(s)){G_L}^i(s)ds\,.$$

is oscillatory or nonnegative, then $y(t)$ *is oscillatory.*

PROOF. Without any loss in generality, we may assume $y(t) > 0$ and in view of (c), $y(g_i(t)) > 0$ for $t \geq t_1$, and $i = 1, 2, \cdots, m$. Let

$$\text{(4)} \qquad x(t) = y(t) - r(t).$$

Then it follows from (1) and (4) that

$$\text{(5)} \qquad x^{(n)}(t) + (-1)^{n+1} \sum_{i=1}^{m} f_i(t, y(t), y(g_i(t))) = 0$$

which implies for $t \geq t$,

$$(-1)^n x^{(n)}(t) > 0. \tag{6}$$

From (d), (4), (6) and $x(t)$ is bounded, there exists a $t_2 \geq t_1$ such that for $t \geq t_2$

$$(-1)^i x^{(i)}(t) \geq 0, \quad i = 1, 2, \dots, n-1. \tag{7}$$

Now by mean value theorem we have

$$\begin{aligned} x(a) &= x(b) + (a-b)x'(b) + \frac{(a-b)^2}{2!}x''(b) \\ &+ \cdots + \frac{(a-b)^{n-1}}{(n-1)!}x^{(n-1)}(b) + \frac{(a-b)^n}{n!}x^{(n)}(\xi) \end{aligned} \tag{8}$$

where $\xi \in (a, b)$. Let $t_2 < s < t$, then by (c) we have $g_i(s) \leq g_i(t)$. Let $a = g_i(s)$, $b = g_i(t)$ in (8) and invoking (7) we have

$$x(g_i(s)) \geq x(g_i(t)) + \frac{(g_i(s)) - g_i(t))^{n-1}}{(n-1)!}x^{(n-1)}(g_i(t)). \tag{9}$$

Multiplying (9) by ${G_L}^i(s)$ and summing up for all $i \in K$, we have by (2)

$$\begin{aligned} &\sum_{i \in k} {G_L}^i(s)x(g_i(t)) + \sum_{i \in K} {G_L}^i(s)\frac{(g_i(s) - g_i(t))^{n-1}}{(n-1)!}x^{(n-1)}(g_i(t)) \\ &\leq \sum_{i \in K} {G_L}^i(s)x(g_i(s)) \leq \sum_{i=1}^{m} f_i(s, y(s), y(g_i(s))) - \sum_{i \in K} r(g_i(s)){G_L}^i(s) \\ &= (-1)^n x^{(n)}(s) - \sum_{i \in K} r(g_i(s)){G_L}^i(s). \end{aligned}$$

Integrating it with respect to s from $g^*(t)$ to t,

$$\begin{aligned} &\sum_{i \in K}(-1)^{n-1}x^{(n-1)}(g_i(t)) \int_{g^*(t)}^{t} (g_i(t) - g_i(s))^{n-1}{G_L}^i(s)ds \\ &\leq (-1)^n x^{(n-1)}(t) - (-1)^n t^{(n-1)}(g^*(t)) - \sum_{i \in K}\int_{g^*(t)}^{t} r(g_i(s)){G_L}^i(s)ds \end{aligned}$$

or

$$\begin{aligned} &(-1)^{n-1}x^{(n-1)}(g^*(t)) \left[\sum_{i \in K}\int_{g^*(t)}^{t} (g_i(t) - g_i(s))^{n-1}{G_L}^i(s)ds - 1\right] \\ &\leq (-1)^n x^{(n-1)}(t) - \sum_{i \in K}\int_{g^*(s)}^{t} r(g_i(s)){G_L}^i(s)ds \end{aligned} \tag{10}$$

Choose T large enough so that $\varphi(T) \geq 0$. Then from (10) we have

$$\varphi(T) + (-1)^{n-1} x^{(n-1)}(g^*(T)) \left[\sum_{i \in K} \int_{g^*(T)}^{T} (g_i(T) - g_i(s))^{n-1} {G_L}^i(s)\, ds - 1 \right] \leq (-1)^n x^{n-1}(T). \tag{11}$$

Thus, by (3), we have a contradiction to the fact that the left-hand side of (11) is nonnegative, while the right-hand side is negative. The proof is now complete.

Remark 1. If L is the common bound of all bounded solutions of (1), as in Theorem 1, ${G_L}^i$ and f_i for $i \in K$ satisfy the conditions (2) and (3), then, by Theorem 1 every bounded solution of (1) is oscillatory.

Example 1. We see easily that

$$y''(t) - 2y(t - \pi) = \cos t$$

has $y(t) = \cos t$ as a bounded oscillatory solution. Here

$$r(t) = -\cos t, \quad G(s) = 2,$$

$$\int_{t-\pi}^{t} r(g(s))G(s)ds = 4 \sin t, \ \liminf_{t \to \infty} \int_{t-\pi}^{t} (t - s)ds = \frac{\pi^2}{4} > 1.$$

Example 2. Consider the following equation

$$y''(t) - 2y(t - \pi) = \sin t$$

which has $y(t) = \sin t$ as a bounded oscillatory solution. Here

$$r(t) = -\sin t, \quad G(s) = 2, \quad \int_{t-\pi}^{t} r(g(s))G(s)ds = 0.$$

Similarly, we can prove the following theorem.

Theorem 2. *Let the conditions* (a), (d) *hold. Assume that for any* $L > 0$, *there exist a nonempty set of indices* K *as in Theorem* 1 *and functions* ${G_L}^i \in C[R^+, R^+]$, $i \in K$ *such that* (2), (3) *hold. Then, every bounded solution of* (1) *is oscillatory.*

Corollary 1. *Assume that* $f_i(t, u, v_i)$ *in* (1) *for* $i = 1, 2, \ldots, m$ *are continuously differentiable with respect to* u *and* v_i, *and*

$$f_i(t, u, v) = \frac{\delta f_i}{\delta u}(t, 0, 0)u + \frac{\delta f_i}{\delta v_i}(t, 0, 0)v_i + F_i(t, u, v_i) \tag{12}$$

with $\sum_{i=1}^{m}\frac{\delta f_i}{\delta u}(t,0,0)$, $\sum_{i=1}^{m}\frac{\delta f_i}{\delta v_i}(t,0,0)$ *are nonnegative and continuous functions on* R^+ *and* $F_i(t,u,v_i)$ *for* $i=1,\ldots,m$ *satisfies the conditions* $(a)-(c)$. *Let*

$$\liminf_{r\to\infty}\sum_{i\in K}\int_{g^*(t)}^{t}(g_i(t)-g_i(s))^{n-1}\frac{\delta f_i}{\delta v_i}(s,0,0)ds\geq 1$$

where $g^*(t)=\max_{i\in K}\{g_i(t)\}$. *Then every bounded solution of* (1) *is oscillatory.*

PROOF. For any $L>0$, set $G^i(t)=\frac{\delta f_i}{\delta v_i}(t,0,0)$ for $i\in K$. Obviously (1) satisfies the condition (2) in view of (12). Thus Theorem 2 implies the conclusion of this corollary is true.

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