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A NOTE ON STARSHAPED SETS

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Abstract. We find the smallest Lipschitz constant for the radial projection of the unit sphere on the boundary of a compact star-shaped set containing the origin in the interior of its convex kernel. We also find the least upper bound of the upper outer surface area in the sense of Minkowski of the boundary of a compact star-shaped set contained in the unit ball and containing a concentric ball of radius a $(0 < a \leq 1)$ in its convex kernel.

1. Introduction. In this note we shall by S, B, Φ denote the unit sphere, the unit ball and the origin of the Euclidean space E_n respectively.

In [5., Problem 3.2.] F. A. Valentine posed the question of finding the smallest Lipschitz constant for the function defined by $f : S \to E_1$, $f(x) = \|\{\lambda x \mid \lambda > 0\} \cap bd K\|$, where K is a compact star-shaped set in Euclidean space E_n whose convex kernel C has a nonempty interior and $\Phi \in \operatorname{int} C$. In [4.] F. A. Toranzos solved this problem. In the second part of this note we shall determine the smallest Lipschitz constant for the function $f : S \to bd K$, $f(x) = \{\lambda x \mid \lambda > 0\} \cap bd K$. Of course, the function f is well defined.

In the third part of this note we shall determine the least upper bound of the upper outer surface area in the sense of Minkowski of a compact star-shaped set contained in the unit ball and containing a concentric ball of radius a $(0 < a \leq 1)$ in its convex kernel. This will give an answer to a question posod by Z. A. Melzak [3., Problem 25]. Another proof of this statement is also given in the special case $n = 2$, using the estimation of Lipschitz constant.

2. THEOREM 1. *Let K be a compact star-shaped set in Euclidean space E_n whose convex kernel C has a nonempty interior and $\Phi \in intC$. Let us denote $M = \max_{x\in K} \|x\|$, and $m = \max\{r \mid K[\Phi, r] \subseteq C\}$. Then the function $f : S \to bd K$, $f(x) = \{\lambda x \mid \lambda > 0\} \cap bd \lambda K$ is Lipschitzian on the unit sphere S and the smallest Lipschitz constant is M^2/m.*

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Proof. Define a function $g : S \times S\backslash\Delta \to R$, $g(x,y) = \|f(x) - f(y)\|/\|x-y\|$. We shall prove that M^2/m is an upper bound for $g(x,y)$. Without lost of generality, we can suppose $\|f(x)\| \geq \|f(y)\|$.

Consider, first, the case when $\sphericalangle x\Phi y = \alpha$, where α is the fixed angle and $0 < \alpha \leq \pi$. Then, $\|x-y\|$ is fixed too.

When the angle $\sphericalangle\Phi f(x)f(y)$ is fixed; $g(x,y)$ has the greatest value when $\|f(x)\| = M$; and when $\|f(x)\|$ is fixed, $g(x,y)$ has the greazest value when the angle $\sphericalangle\Phi f(x)f(y)$ attains one of its extremal values.

a) $[f(x),\ f(y)] \cap K[\Phi,m] \neq \emptyset$

Let $z = \{\lambda f(y) \mid \lambda > 0\} \cap S(\Phi,m)$.

The relation $\sphericalangle\Phi z f(x) \leq \pi/2$ holds, so that $g(x,y)$ has the greatest value when the angle $\sphericalangle\Phi f(x)f(y)$ attains its maximal value, i.e. when $\|f(y)\| = \|f(x)\|$ and then we have

$$g(x,y) = \|f(x)\| \leq M \leq M^2/m$$

b) $[f(x),\ f(y)] \cap K[\Phi,m] = \emptyset \Rightarrow \text{arc sin}(m/M) \leq \sphericalangle\Phi f(x)f(y) \leq \frac{\pi-\alpha}{2}$

(i) $\sphericalangle\Phi f(x)f(y) = (\pi-\alpha)/2 \Rightarrow g(x,y) = \|f(x)\| \leq M \leq M^2/m$

(ii) $\sphericalangle\Phi f(x)f(y) = \text{arc sin}(m/M)$

Let us denote $u = f(x)f(y) \cap K[\Phi,m]$ and $\beta = \sphericalangle f(x)\Phi u = \text{arc cos}\,\frac{m}{M}$. Then we have

$$\begin{aligned} g(x,y) &= \frac{\sqrt{M^2-m^2} - m\,\mathrm{tg}\,(\beta-\alpha)}{2\sin(\alpha/2)} \\ &= \frac{\sqrt{M^2-m^2} - m(\sqrt{M^2-m^2} - m\,\mathrm{tg}\,\alpha)/(m+\sqrt{M^2-m^2}\,\mathrm{tg}\,\alpha)}{2\sin(\alpha/2)} \\ &= \frac{M^2\mathrm{tg}\,\alpha}{2\sin(\alpha/2)\cdot(m+\sqrt{M^2-m^2}\cdot\mathrm{tg}\,\alpha)} \\ &= \frac{M^2\sqrt{1+\mathrm{tg}^2(\alpha/2)}}{m+2\sqrt{M^2-m^2}\cdot\mathrm{tg}\,(\alpha/2) - m\cdot\mathrm{tg}^2(\alpha/2)} \end{aligned}$$

Put $\mathrm{tg}\,(\alpha/2) = t(0 < t \leq \sqrt{M^2-m^2}/m)$

$$g(x,y) = \frac{M^2\sqrt{1+t^2}}{M + 2\cdot\sqrt{M^2-m^2}\cdot t - m\cdot t^2} = g_0(t)$$

$$g_0'(t) = M^2 m\frac{t^3+3t-2\sqrt{M^2-m^2}/m}{\sqrt{1+t^2}(m+2\sqrt{M^2-m^2}\cdot t - m\cdot t^2)^2}$$

$$h(t) \equiv t^3 + 3\cdot t - 2\cdot\sqrt{M^2-m^2}/m \Rightarrow \text{sign}\,g_0'(t) = \text{sign}\,h(t)$$

We have $h'(t) = 3t^2 + 3 > 0$ and $h(0) < 0$.

There is a unique point $t_0 > 0$ such that $h(t_0) = 0$. On $(0, t_0) g_0$ is decreasing and on $(t_0, \sqrt{M^2 - m^2}/m) g_0$ is increasing. The greatest value of $g_0(t)$, for $0 < t \leq \sqrt{M^2 - m^2}/m$, is then $\lim_{t \to 0_+} g_0(t) = g_0(0)$ or $g_0(\sqrt{M^2 - m^2}/m)$. But,

$$g_0(0) = M^2/m \qquad \text{and} \qquad g_0(\sqrt{M^2 - m^2}/m) = M \leq M^2/m.$$

Hence we have proved that $g(x, y) \leq M^2/m$ holds, for arbitrary angle $\alpha(0 < \alpha \leq \pi)$. Hence we have

$$\|f(x) - f(y)\| \leq (M^2/m) \cdot \|x - y\|$$

From the given proof it is easy to see that $\sup\{g(x, y) \mid x, y \in S, x \neq y\} = M^2/m$, when bdK contains a segment $[x, y]$ of line p such that $\|x\| = M$ and $d(\Phi, p) = m$. In particular, such an example is the following $K = \operatorname{conv}(K[\Phi, m] \cup \{x\})$, where $\|x\| = M > m$.

3. In [3.] Z. A. Melzak posed the question of finding the least upper bound of the surface area of the boundaiy of a compact star-shaped set $K \subset E_n$ contained in a ball of radius 1 and containing a concentric ball of radius a $(0 < a \leq 1)$ in its convex kernel. In this note we shall determine the least upper bound of the upper outer surface area in the sense of Minkowski of the boundary of such sets.

The upper outer surface area in the sense of Minkowski is defined by

$$\bar{A}_+(K) = \lim_{\varepsilon \to 0_+} (V(K + \varepsilon B) - V(K))/\varepsilon$$

and the surface area in the sense of Minkowski is

$$A(K) = \lim_{\varepsilon \to 0} (V(K + \varepsilon B) - V(K))/\varepsilon,$$

if this limit exists. If there exists the surface area in the sense of Minkowski of the boundary of some set, then it, obviously, coincides with the upper outer surface area of the boundary of this set. In the case of a polytope (not necessarly convex) or a convex set, this limit exists and the surface area in the sense of Minkowski coincides with a usual meaning of the surface area (see [2]). So, it is enough to determine the least upper bound of the upper outer surface area in the sense of Minkowski of those sets.

THEOREM 2. *Let B_1 and B_2 be two concentric balls in Euclidean space E_n, B_1 of radius* 1 *and B_2 of radius a $(0 < a \leq 1)$. A compact set K is contained in B_1 and is starshaped at every point of B_2. Then the least upper bound of the upper outer surface area in the sense of Minkowski of the boundary of K is $A(B)/a$.*

Proof. Without lost of generality, we can suppose that B_1 is the unit ball B. First, we shall prove the inclusion $K + \varepsilon B \subseteq (1 + \varepsilon/a)K$. Let $x \in (K + \varepsilon B) \backslash K$ and $[\Phi, x] \cap bdK = \{y\}$, and $z \in K$ such that $d(z, x) \leq \varepsilon$. Let us denote with p the line through y and z. Then $p \cap \operatorname{int} B_2 = \emptyset$, since y is the boundary point of K, and so $d(\Phi, p) \geq a$. Also $d(x,p) \leq d(x,z) \leq \varepsilon$. Hence, we have

$$d(x,y) = \frac{d(x,p)}{d(\Phi,p)} d(\Phi,y) \leq \frac{\varepsilon}{a} d(\Phi,y)$$
$$d(\Phi,x) \leq \left(1 + \frac{\varepsilon}{a}\right) d(\Phi,y)$$
$$K + \varepsilon B \subseteq \left(1 + \frac{\varepsilon}{a}\right) K$$

Using the well known equality $nV(B) = A(B)$ we get

$$\begin{aligned}\bar{A}_+(K) &= \overline{\lim_{\varepsilon\to 0_+}} \frac{V(K+\varepsilon B) - V(K)}{\varepsilon} \leq \overline{\lim_{\varepsilon\to 0_+}} \frac{((1+\varepsilon/a)^n - 1)V(K)}{\varepsilon} \\ &= \frac{nV(K)}{a} \leq \frac{nV(B)}{a} = \frac{A(B)}{a}.\end{aligned}$$

Now, we shall prove, that there is a polytope with the surface area arbitrary close to $A(B)/a$, satisfying the conditions of the theorem, what will end the proof. We can take here the surface area of the boundary of a polytope and of sphere, which is the boundary of the convex set, in its usual meaning, because it coincides with the surface area in the sense of Minkowski for these sets. Also, we shall take the same symbol for these two meanings of the surface area. So, for example, $A(B)$ and $A(S)$ both denote the surface area of the unit sphere.

Let F_δ be a finite δ-net for a sphere S. For every point $p \in F_\delta$, let Δ_p be a simplex with p as a vertex and such that the faces which have p as a vertex are congruent one to each other, and the hyperplanes of those faces support B_2. Let us choose other vertices of Δ_p to be the nearest points to the origin Φ on the lines which ate the intersections of those supporting hyperplanes of B_2. Then, for every real positive number ε, there is $\delta > 0$ such that $K_\varepsilon = B_{1-\varepsilon} \cup (\cup\{\Delta_p \mid p \in F_\delta\})$ is a polytope whose each face containing p is contained in a hyperplane supporting B_2 and such that $B_{1-\varepsilon} \subseteq K_\varepsilon \subseteq B$.

Let L be a triangulated boundary of K_ε. For each simplex V of L let V' be a simplex whose vertices are thc radial projections of the vertices of V on S, and let K' be a set bounded by those simplices. For each simplex V in L the inequality

$$A(V) \geq (1-\varepsilon)^{n-1} \frac{1}{\cos\varphi} A(V'),$$

holds, where φ is the angle between the hyperplanes of V and V'. Then, the angle between the perpendiculars on those hyperplanes through the origin is also φ and we have

$$A(V) \geq (1-\varepsilon)^{n-1} \frac{1-\varepsilon}{a} A(V') = \frac{(1-\varepsilon)^n}{a} A(V')$$
$$A(K_\varepsilon) \geq (1-\varepsilon)^n \frac{1}{a} A(K') \geq (1-\varepsilon)^n \frac{1}{a} A(\operatorname{conv} K') \to A(B)/a, \quad \varepsilon \to 0.$$

So, for ε small enough, $A(K_\varepsilon)$ is arbitrary close to $A(B)/a$.

Remark. In the same way we can prove, for $\varepsilon > 0$, $K - \varepsilon B \supseteq (1-\varepsilon/a)K$, where $K - \varepsilon B = \{x \in K \mid d(x, bd\,K) \geq \varepsilon\}$. So, we have also

$$\bar{A}_-(K) = \overline{\lim_{\varepsilon\to 0_-}}(V(K+\varepsilon B) - V(K))/\varepsilon \leq A(B)/a$$

Hence, $A(B)/a$ is also the least upper bound of the upper inner surface area in the sense of Minkowski of $bd\,K$.

Applying Theorem 1. we shall give another proof of the above statement in the special case $n = 2$. This time, we shall use usual definition of a length of a curve in E_2.

Proposition 1. Let B_1 and B_2 be two concentric solid circles in Euclidean plane E_2, B_1 of radius 1 and B_2 of radius a $(0 < a \leq 1)$. A compact set K is contained in B_1 and is star-shaped at every point of B_2. Then the least upper bound of the length of the boundary of K is $2\pi/a$.

Proof. Without lost of generality, we can suppose that the origin is the center of the solid circles B_1 and B_2. Let $x_1, x_2, \ldots, x_n \in bd\,K$ be the vertices of a closed, polygonal line in succesive order, i.e. such that the angles $\sphericalangle x_1, \Phi x_2, \sphericalangle x_2 \Phi x_3, \ldots, \sphericalangle x_n \Phi x_1$ have the same orientation. Let f be as in Theorem 1. defined function. Then the points $f^{-1}(x_1), f^{-1}(x_2), \ldots, f^{-1}(x_n)$ are the vertices of the closed, convex, polygonal line inscribed in B_1. Then by Theorem 1. we have

$$\begin{aligned}
&\|x_2 - x_1\| + \|x_3 - x_2\| + \cdots + \|x_n - x_{n-1}\| + \|x_1 - x_n\| \\
&= \|f(f^{-1}(x_2)) - f(f^{-1}(x_1))\| + \cdots + \|f(f^{-1}(x_1)) - f(f^{-1}(x_n))\| \\
&\leq (\|f^{-1}(x_2) - f^{-1}(x_1)\| + \cdots + \|f^{-1}(x_1) - f^{-1}(x_n)\|)/a \leq 2\pi/a
\end{aligned}$$

Let $x_1, x_2, \ldots, x_n$ be now the vertices of a regular n-gon inscribed in S. Let us put

$$K_n = \operatorname{con} v(\{x_1\} \cup B_2) \cup \operatorname{con} v(\{x_2\} \cup B_2) \cup \cdots \cup \operatorname{con} v(\{x_n\} \cup B_2).$$

Then the boundary of K_n is polygonal line with $2n$ equal line segments. Let one of these segments is $[x_1, y_1]$. Then

$$\|x_1 - y_1\| = \sqrt{1-a^2} - a\frac{\sqrt{1-a^2}/a - \operatorname{tg}(\pi/n)}{1 + (\sqrt{1-a^2}/a)\cdot \operatorname{tg}(\pi/n)} = \frac{\operatorname{tg}(\pi/n)}{a + \sqrt{1-a^2}\operatorname{tg}(\pi/n)}$$

Then, the length of this polygonal line is

$$\begin{aligned}
&p_n = 2n\frac{\operatorname{tg}(\pi/n)}{a + \sqrt{1-a^2}\cdot \operatorname{tg}(\pi/n)} = 2\pi/a\frac{\operatorname{tg}(\pi/n)}{\pi/n}\frac{1}{1 + (\sqrt{1-a^2}/a)\cdot \operatorname{tg}(\pi/n)} \\
&p_n \to 2\pi/a, \quad n \to \infty
\end{aligned}$$

Hence, K_n is a sequence of compact sets satisfying the conditions of the proposition, whose length of the boundary converges to $2\pi/a$. So, $2\pi/a$ is the least upper bound of the length of the boundary of such sets.

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