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# THE REES CONGRUENCE IN UNIVERSAL ALGEBRAS

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Dedicated to my teacher Univ. Doz. Dr. H. Mitsch

**Abstract**. In the famous semigroup-theoretical paper of D. Rees (1940) the author introduced a special congruence for semigroups and proved some fundamental isomorphism theorems. We generalize the ideas of Rees to universal algebras and introduce the notion of Rees congruence in a universal algebra. We prove an isomorphism theorem and apply it to study the notion of Rees series in universal algebras; this is a generalization of the notion of normal series in group theory. Further we study the structure of the set of all Rees congrunces and in the last part of the paper we present some applications of this theory to algebras of polynomial functions.

## 1. Introduction

Let $A$ be a nonempty set and $\Omega = \{w_i: i < o,\ o \text{ an ordinal number}\}$ a nonempty set of $n_i$-ary operations on $A$. Then $\mathfrak{A} = (A, \Omega)$ is called a universal algebra. We introduce a notion of a special of subalgebra of $\mathfrak{A}$.

DEFINITION 1.1. Let $M \subseteq A$ be a subalgebra of $\mathfrak{A}$, and for every $n$-ary operation $w \in \Omega$ and every $(a, b) \in M^2$ and $x_1, \dots, x_{n-1} \in A$ the following condition might hold:

$$(*)\begin{cases} \omega(a, x_1, \dots, x_{n-1}),\ \omega(b, x_1, \dots, x_{n-1}) \in M \ \ \textit{or} \\ \qquad\qquad \omega(a, x_1, \dots, x_{n-1}) = \omega(b, x_1, \dots, x_{n-1}) \\ \omega(x_1, a, \dots, x_{n-1}),\ \omega(x_1, b, \dots, x_{n-1}) \in M \ \ \textit{or} \\ \qquad\qquad \omega(x_1, a, \dots, x_{n-1}) = \omega(x_1, b, \dots, x_{n-1}) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \omega(x_1, \dots, x_{n-1}, a),\ \omega(x_1, \dots, x_{n-1}, b) \in M \ \ \textit{or} \\ \qquad\qquad \omega(x_1, \dots, x_{n-1}, a) = \omega(x_1, \dots, x_{n-1}, b) \end{cases}$$

Then $M$ is called a Rees subalgebra of $\mathfrak{A} = (A, \Omega)$.

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In the following proposition we generalize the wellknown semigroup-theoretical notion of Rees congruence to universal algebras.

PROPOSITION 1.2. *Let $M$ be a Rees subalgebra of $\mathfrak{A} = (A, \Omega)$ and $\rho_M$ the following binary relation on $A$: $x \equiv y(\rho_M)$: $x, y \in M$ or $x = y$. Then $\rho_M$ is a congruence on $\mathfrak{A}$.*

*If $\rho$ is a congruence on $\mathfrak{A}$ with at most one non-singleton class $M$ and $M$ is a subalgebra of $\mathfrak{A}$, then $M$ is a Rees subalgebra and $\rho = \rho_M$.*

The proof of this proposition is obvious.

REMARK 1.3. If the algebra $\mathfrak{A} = (S, \cdot)$ is a semigroup, then every semigroup ideal $I$ is a Rees subalgebra. But there are Rees subalgebras that are not ideals of $S$. Let $S$ be a semigroup without null and $S^* = S \cup \{0\}$ with $a \cdot 0 = 0 \cdot a = 0$ for every $a \in S^*$, then $S$ is a Rees subalgebra of $S^*$ but not a semigroup ideal.

In Szasz (1968) Rees congruences on lattices $(L, \wedge, \vee)$ are studied; the author proved, Theorem 1., p. 261, that an ideal $I$ is a Rees subalgebra, if the following condition holds:

$$a \in L - I, \qquad b \in I: \qquad a \geq b.$$

## 2. General theory of Rees congruence

In this chapter we study for an arbitrary Rees subalgebra $M$ the Rees congruence $\rho_M$. First we prove an isomorphism theorem that formally looks like the second isomorphism theorem in group theory.

PROPOSITION 2.1. *If $M \subseteq A$ is a Rees subalgebra and $T \subseteq A$ is a subalgebra of $\mathfrak{A}$ such that $M \cap T \neq \emptyset$, then $M \cup T$ is a subalgebra of $\mathfrak{A}$, too.*

PROOF: Obviously, nullary and unary operations preserve an arbitrary union of subalgebras. Let $\omega$ be an $n$-ary operation, $n \geq 2$, and $\mathbf{a} = (a_1, \ldots a_n) \in (M \cup T)^n$, $b \in M \cup T \neq \emptyset$. If $\mathbf{a} \in T^n$, then $\omega(\mathbf{a}) \in T \subseteq M \cup T$. If $\mathbf{a} \notin T^n$, then let $\mathbf{a}'$ denote the vector obtained from $\mathbf{a}$ by substituting all its components from $M$ with $b$. Then $\mathbf{a}' \equiv \mathbf{a}(\rho_M)$ componentwise, but $\mathbf{a}' \in T^n$, hence $\omega(\mathbf{a}') \in T$. Because of $\mathbf{a} \equiv \mathbf{a}(\rho_M)$ we have $\omega(\mathbf{a}) = \omega(\mathbf{a}')$ or $\omega(\mathbf{a})$, $\omega(\mathbf{a}') \in M$. In the first case we obtain $\omega(\mathbf{a}) = \omega(\mathbf{a}') \in T$ and in the second case $\omega(\mathbf{a}) \in M$; and so in both cases $\omega(\mathbf{a}) \in M \cup T$ and the proposition is proved.

THEOREM 2.2. *Let $M$ be a Rees subalgebra of $\mathfrak{A} = (A, \Omega)$ and $T \subseteq A$ a subalgebra of $\mathfrak{A}$ such that $M \cap T \neq \emptyset$. If $A/M$ denotes the Rees factor algebra, of $\mathfrak{A}$ with respect to the congruence $\rho_M$, then the following isomorphism can be stated:*

$$T/ \cup T \cong M \cup T/_M.$$

PROOF: Under the condition of this theorem it is clear that $\rho_{M \cap T}$ is a congruence on $T$, the restruction of $\rho_M$ on $T$, and similary $\rho_M$ is a congruence on $M \cup T$. Now obviously the two algebras $T/_{M \cap T}$ and $M \cup T/_M$ are isomorphic.

Definition 2.3. Let $\mathfrak{A} = (A, \Omega)$ be a universal algebra. Then a Rees series of $\mathfrak{A}$ is a finite descending series

$$A = A_0 \supset A_1 \supset \ldots \supset A_r = \emptyset$$

such that $A_0, \ldots, A_{r-1}$ are Rees subalgebras of $A$, $A_r = \emptyset$ and $A_{i+1}$ is properly contained in the algebra $A_i$. We define the factors of the Rees series to be the Rees factor algebras $A_i/A_{i+1}$ for $i = 0, \ldots, r-1$. The length of the Rees series is the number of factors. A refinement of a Rees series is any Rees series containing every term $A_i$ of the orginal series. Two Rees series are termed isomorphic if we can put their factors in $1-1$ correspondence so that corresponding factors are isomorphic. Further we define a composition series to be a Rees series without proper refinements.

Proposition 2.4. *Let $\mathfrak{A} = (A, \Omega)$ be a universal algebra and $S$, $T$ Rees subalgebras of $\mathfrak{A}$ such that $S \cap T \neq \emptyset$. Then $S \cup T$ is a Rees subalgebra, too.*

Proof: Because of Proposition 2.1. $S \cup T$ is a subalgebra of $\mathfrak{A}$. Let $\omega$ be an $n$-ary operation of $\mathfrak{A}(n \geq 2)$ and $x_1, \ldots, x_{n-1} \in A$ and $(a,b) \in (S \cup T)^2$ and $d \in S \cap T \neq \emptyset$. If $a, b \in S$ we obtain

$$\begin{aligned}
\omega(a, x_1, \ldots, x_{n-1}),\ \omega(b, x_1, \ldots, x_{n-1}) &\in S \text{ or} \\
\omega(a, x_1, \ldots, x_{n-1}) &= \omega(b, x_1, \ldots, x_{n-1}); \text{ hence} \\
\omega(a, x_1, \ldots, x_{n-1}),\ \omega(b, x_1, \ldots, x_{n-1}) &\in S \cup T \text{ or} \\
\omega(a, x_1, \ldots, x_{n-1}) &= \omega(b, x_1, \ldots, x_{n-1}).
\end{aligned}$$

If $a, b \in T$ this results similary. Now let $a \in S$, $b \in T$; we obtain $a \equiv d(\rho_S)$ and $b \equiv d(\rho_T)$. $\rho_S$ and $\rho_T$ are congruences, $\rho_{S \cup T}$ is an equivalence relation such that $\rho_S$, $\rho_T \leq \rho_{S \cup T}$; therefore

$$\begin{aligned}
\omega(a, x_1, \ldots, x_{n-1}) &\equiv \omega(d, x_1, \ldots, x_{n-1}) \quad (\rho_S) \\
\omega(b, x_1, \ldots, x_{n-1}) &\equiv \omega(d, x_1, \ldots, x_{n-1}) \quad (\rho_T),
\end{aligned}$$

and so

$$\omega(a, x_1, \ldots, x_{n-1}) \equiv \omega(b, x_1, \ldots, x_{n-1}) \quad (\rho_{S \cup T}).$$

Hence

$$\begin{gathered}
\omega(a, x_1, \ldots, x_{n-1}),\ \omega(b, x_1, \ldots, x_{n-1}) \in S \cup T \quad \text{or} \\
\omega(a, x_1, \ldots, x_{n-1}) = \omega(b, x_1, \ldots, x_{n-1}).
\end{gathered}$$

If $a \in T$, $b \in S$ this results similary. So condition $(^*)$ of Definition 1.1. can be be verified, therefore $S \cup T$ is a Rees subalgebra of $\mathfrak{A}$.

REMARK 2.5. We consider two groupids $G_1$ and $G_2$ defined by the following multiplication tables:

| $G_1$ | $0$ | $a$ | $b$ |
|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $0$ |
| $b$ | $0$ | $0$ | $b$ |

| $G_2$ | $0$ | $a$ | $e$ | $f$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $0$ | $e$ |
| $e$ | $0$ | $0$ | $e$ | $e$ |
| $f$ | $0$ | $e$ | $e$ | $f$ |

The groupoid $G_1$ shows that we cannot omit the condition $S \cap T \neq \emptyset$ for the Rees subalgebras in Proposition 2.4., because $S = \{a\}$ and $T = \{b\}$ are Rees subalgebras of $G_1$ with $S \cap T \neq \emptyset$, but $S \cup T = \{a, b\}$ is not a Rees subalgebra of $G_1$.

We write $S \underset{R}{<} T$, if $S$ is a Rees subalgebra of $T$. Then the binary relation $\underset{R}{<}$ is not transitive in general. We take the groupoid $G_2$ and put $S = \{e, f\}$ and $T = \{0, e, f\}$; then $S \underset{R}{<} T$, $T \underset{R}{<} G_2$, but $S$ is not a Rees subalgebra of $G_2$.

LEMMA 2.6. *Let $\mathfrak{A} = (A, \Omega)$ be a universal algebra and $R$, $S$ two subalgebras and $R^*$, $S^*$ be Rees subalgebras of $A$ such that $R^* \cap S^* \neq \emptyset$. Write*

$$T = R^* \cup (R \cap S), \qquad T^* = R^* \cup (R \cap S^*)$$
$$U = S^* \cup (R \cap S), \qquad U^* = S^* \cup (R^* \cap S).$$

*Then $T^*$, $U^*$ are Rees subalgebras of $T$, $U$, respectively, and*

$$T/_{T^*} \cong U/_{U^*}$$

PROOF: Because of the condition $R^* \cap S^* \neq \emptyset$ and Proposition 2.1. $T = R^* \cup (R \cap S)$ is a subalgebra $T^* = R^* \cup (R \cap S^*)$ is a Rees subalgebra of $T$; since $R \cap S^* \underset{R}{<} A$, $R^* \underset{R}{<} A$, we obtain $R^* \cup (R \cap S^*) \underset{R}{<} A$ by Proposition 2.4.; therefore $T^* \underset{R}{<} T$, because $T \subseteq A$. Similary we obtain that $U^*$ is a Rees subalgebra of $U$.

We obtain

$$T^* \cup (R \cap S) = (R^* \cup (R \cap S^*)) \cup (R \cap S) = R^* \cup (R \cap S) = T.$$

Hence by Theorem 2.2.,

$$T/_{T^*} \cong R \cap S/_{T^* \cap (R \cap S)}.$$

But

$$(R \cap S) \cap T^* = (R^* \cap S) \cup (R \cap S^*),$$

hence

$$T/_{T^*} \cong R \cap S/_{(R^* \cap S) \cup (R \cap S^*)}.$$

$U$, $U^*$ are obtained from $T$, $T^*$ by interchanging $R$, $R^*$ with $S$, $S^*$. Hence as above we may prove that

$$U/_{U^*} \cong R \cap S/_{(R^* \cap S) \cup (R \cap S^*)};$$

and so

$$T/_{T^*} \cong U/_{U^*}.$$

THEOREM 2.7. *Let $\mathfrak{A} = (A, \Omega)$ be a universal algebra and*

$$A = S_0 \supset S_1 \supset \ldots \supset S_r = \emptyset$$
$$A = T_0 \supset T_1 \supset \ldots \supset T_s = \emptyset$$

*two Rees series with $S_i \cap T_j \neq \emptyset$ for $i = 0, \ldots, r-1$ and $j = 0, \ldots, s-1$. Then the two series have isomorphic refinements.*

PROOF: The required rafinements are, respectively,

$$A = S_{00} \supset S_{01} \supset \ldots \supset S_{0s} = S_{10} \supset \ldots \supset S_{rs}$$
$$A = T_{00} \supset T_{10} \supset \ldots \supset T_{r0} = T_{01} \supset \ldots \supset T_{rs},$$

where

$$S_{ik} = S_{i+1} \cup (S_i \cap T_k), \qquad T_{ik} = T_{k+1} \cup (T_k \cap S).$$

Define $R^* = S_{i+1}$, $R = S_i$, $S^* = T_{k+1}$, $S = T_k$; then $R^*$ is a Rees subalgebra of $R$ and $S^*$ is a Rees subalgebra of $S$ and $R^* \cap S^* \neq \emptyset$ $(i < r; k < s)$. So we can apply Lemma 2.5. and obtain:

$$S_{ik}/_{S_{ik+1}} \cong T_{ik}/_{T_{i+1k}},$$

and so the above refinements are isomorphic.

COROLLARY 2.8. *Let $\mathfrak{A} = (A, \Omega)$ be a universal algebra, such that $S \cap T \neq \emptyset$ for all subalgebras $S$, $T$ of $\mathfrak{A}$. Then all composition series of $\mathfrak{A}$ are isomorphic.*

REMARK 2.9. The condition $S \cap T \neq \emptyset$, for all subalgebras $S$, $T$, holds if $\mathfrak{A}$ contains nullary operations.

In the following let $R(\mathfrak{A})$ denote the set of all Rees subalgebras of $\mathfrak{A}$ and the empty set $\emptyset$. Then the inclusion $\subseteq$ defines a lattice ordering on $R(\mathfrak{A})$ with the operations $\wedge$, $\vee$ defined by

$$M \wedge N = \inf(M, N) = M \cap N$$
$$M \vee N = \sup(M, N) \quad \text{for} \quad M, N \in R(\mathfrak{A}).$$

For every subset $T \subseteq A$ we introduce the Rees subalgebra $\langle T \rangle$ generated by $T$; this is the algebra $\langle T \rangle = \cap_{T \subseteq M \in R(\mathfrak{A})}$.

A Rees subalgebra $M$ is called finetely generated in $R(\mathfrak{A})$, if a finite set $T = \{a_1, \ldots, a_n\}$ exists such that $M = \langle T \rangle$.

Now every $n$-ary operation $\omega \in \Omega$ defines an $n$-ary operation $\omega^*$ on $R(\mathfrak{A})$ by

$$\omega^*(M_1, \dots, M_n) = \langle \{\omega(x_1, \dots, x_n) \colon x_1 \in M_1, \dots, x_n \in M_n\} \rangle \quad \text{for } M_1, \dots, M_n \in R(\mathfrak{A}).$$

$\Omega^*$ denotes the set of all such operations $\omega^*$.

PROPOSITION 2.10. *If $\mathfrak{A} = (A, \Omega)$ is a universal algebra, then $(R(\mathfrak{A}), \Omega^*, \subseteq)$ is a lattice-ordered algebra: so*

$$\omega^*(M_1, \dots, M_n) \subseteq \omega^*(N_1, \dots, N_n)$$

*for $M_1 \subseteq N_1, \dots, M_n \subseteq N_n$ and all operations $\omega^* \in \Omega^*$.*

| $G_3$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $a$ | $c$ | $b$ |
| $b$ | $c$ | $b$ | $a$ |
| $c$ | $b$ | $a$ | $c$ |

If $S \cap T \neq \emptyset$ fon all Rees subalgebras $S$, $T$, the lattice $R(\mathfrak{A})$ is distributive; generally this does not hold (see the groupid $G_3$).

THEOREM 2.11. *Let $\mathfrak{A} = (A, \Omega)$ denote a universal algebra and $S$ a proper Rees subalgebra of $\mathfrak{A}$. If $A$ is finitely generated in $R(\mathfrak{A})$, then a maximal proper Rees subalgebra $M$ of $\mathfrak{A}$ exists such that $M \supseteq S$.*

PROOF: Let $\mathfrak{M}$ denote the set of all proper Rees subalgebras $T \supseteq S$ and let $\mathfrak{N}$ be a totally ordered subset of $\mathfrak{M}$. We prove $V = \cup_{T \in \mathfrak{n}} T \in \mathfrak{M}$. Let $\omega$ be an $n$-ary operation and $x_1, \dots, x_n \in V$, then a Rees subalgebra $T_0 \supseteq S$ exists such that $x_1, \dots, x_n \in T_0$, so $\omega(x_1, \dots, x_n) \in T_0 \subseteq V$ and $V$ is a subalgebra. If $a, b \in V$ and $x_1, \dots, x_{n-1} \in A$, then a Rees subalgebra $T_1 \supseteq S$ exists, such that $a, b \in T_1$ and the condition $(*)$ of Definition 1.1. is valid; so $V$ is a Rees subalgebra of $\mathfrak{A}$.

We show $V \neq A$. $A$ is finitely generated by the set $\{a_1, \dots, a_k\}$. If we suppose $V = A$, we obtain $a_1, \dots, a_k \in V$; therefore a Rees subalgebra $T_2 \in \mathfrak{M}$ exists with $a_1, \dots, a_k \in T_2$ and so $T_2 = A$, a contradiction to $T_2 \in \mathfrak{M}$. So we obtain $V \neq A$ and $V \in \mathfrak{M}$ is proved. Now we apply Zorn's Lemma, so $\mathfrak{M}$ contains a maximal element and the theorem is proved.

In the following we say that the maximal chain-condition holds in $R(\mathfrak{A})$, if every increasing chain $S_0 \subset S_1 \subset S_2 \subset \dots$ of Rees subalgebras of $\mathfrak{A}$ contains only a finite number of terms $S_i$. Similary we say that the minimal chain-condition holds in $R(\mathfrak{A})$, if every descending chain $S_0 \supset S_i \supset \dots$ of Rees subalgebras of $\mathfrak{A}$ contains only a finite number of terms $S_i$.

THEOREM 1.12. *Let $\mathfrak{A} = (A, \Omega)$ be a universal algebra. Then all Rees subalgebras $S$ of $\mathfrak{A}$ are finitely generated in $R(\mathfrak{A})$, if and only if the maximal chain-condition holds in $R(\mathfrak{A})$.*

PROOF: First we assume that the maximal chain-condition holds. Let $S$ be an arbitrary Rees subalgebra and $a_1 \in S$. If $S = \langle a_1 \rangle$, then $S$ is finitely generated; if $a_2 \in S$ exists with $a_2 \notin \langle a_1 \rangle$, then $\langle a_1 \rangle \subset \langle a_1, a_2 \rangle$. So we construct an increasing chain of Rees subalgebras

$$\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \langle a_1, a_2, a_3 \rangle \subset \dots ,$$

which must bo finite because of the maximal chain-condition; so $S$ is finitely generated in $R(\mathfrak{A})$.

Now we assume that every Rees subalgebra $S$ is finitely generated in $R(\mathfrak{A})$ and let $S_1 \subset S_2 \subset S_3 \subset \dots,$ be an increasing chain of Rees subalgebras. The union $S = \cup S_i$ of all those Rees subalgebras $S_i$ is finitely generated because of our assumption; so $S = \langle a_1, \dots , a_k \rangle$. Therefore a Rees subalgebra $S_n$ exists such that $a_1, \dots , a_k \in S_n$; hence $S_n = S$. So we have proved that $S = S_n = S_{n+1} = \dots$ and the maximal chain-condition is valid in $R(\mathfrak{A})$.

THEOREM 2.13. *Let $\mathfrak{A} = (A, \Omega)$ denote a universal algebra such that for all subalgebras $S$, $T$ the condition $S \cap T \neq \emptyset$ holds. Then a necesary and sufficient condition for $\mathfrak{A}$ to have a composition series is that the maximal and minimal chain-condition hold in $R(\mathfrak{A})$.*

PROOF: First we assume that a composition series with length $k$ exists. If the maximal or minimal chain-condition does not hold we easily obtain a Rees series with length $n > k$, a contradiction, because we can apply Theorem 2.7. (for all $S, T$ with $S \cap T \neq \emptyset$). If both chain-conditions hold, we construct a composition series as follows. Define $S_0 = A$ and $S_1$ a maximal Rees subalgebra with $S_1 \subset S_0$; such $S_1$ exists because of the maximal chain-condition. Then $S_2$ is a maximal Rees subalgebra of $S_1$ and, generally, $S_n$ a maximal subalgebra of $S_{n-1}$. So we obtain a Rees series

$$A = S_0 \supset S_1 \supset S_2 \dots \supset S_{n-1} \supset S_n \supset \dots ,$$

which must be finite because of the minimal chain-condition. Obviously, this Rees series is a composition series.

EXAMPLE 3.14. Let $(N, +, \cdot)$ be the natural numbers with addition and multiplication; then all Rees subalgebras are of the form

$$S_\alpha = \{x \in \mathbf{N} : x \geq \alpha\}.$$

Then $S \cap T \neq \emptyset$ for all subalgebras $S$, $T$ of $(N, +, \cdot)$; the maximal chain-condition holds in $R(\mathbf{N})$ and the minimal chain-condition does not hold. So $\mathbf{N}$ is finitely generated in $R(\mathbf{N})$ ($\mathbf{N} = \langle 1 \rangle$) and $\mathbf{N}$ does not contain a composition series.

EXAMPLE 2.15. Let $\gamma = (A, \cdot, e)$ be a monoid; we define an $n$-ary operation $\omega$ on $A$ by $\omega(x_1, \dots , x_n) = x_1 \cdot \dots \cdot x_n$. Then $\mathfrak{A} = (A, \omega, e)$ is an algebra with $R(\mathfrak{A}) = R(\gamma)$.

Definition 2.16. Let $\omega$ be an $n$-ary operation of the algebra $\mathfrak{A} = (A, \Omega)$. Then an element $a \in A$ is called idempotent with respect to $\omega$, if and only if $\omega(a, \dots, a) = a$; an element $a \in A$ is called idempotent with respect to all $n$-ary operations $(n \geq 1)$ of $\mathfrak{A}$.

An algebra is called idempotent, if and only if all its elements are idempotent.

Proposition 2.17. *Let $\mathfrak{A} = (A, \Omega)$ be an idempotent algebra and $\rho$ a congruence on $\mathfrak{A}$ with at most one non-singleton class $M$. If $M$ contains all nullary operations of $\mathfrak{A}$, $\rho$ is a Rees congruence.*

Proof: Using Proposition 1.2, we have to prove that the non-singleton class $M$ is a subalgebra of $\mathfrak{A}$. Let $a_j \in M$ $(j = 1, \dots, n)$ and $\omega$ be an $n$-ary operation of $\mathfrak{A}$. Then $a_1 \equiv a_2 \equiv \cdots \equiv a_n(\rho)$, because $\rho$ is a congruence. Since $a_1$ is idempotent, $\omega(a_1, \dots, a_1) = a_1 \in M$ results; so we obtain $\omega(a_1, \dots, a_n) \in M$ and the proposition is proved.

Remark 2.18. Proposition 2.17. does not hold for an arbitrary algebra $\mathfrak{A}$. If we take the free semigroup $(A, \cdot)$ with $A = \{a, a^2, a^3, \dots\}$, the identity relation $\rho$ is a congruence with at most one non-singleton class, but it is not a Rees congruence.

Proposition 2.19. *Let $u = (A, \Omega)$ be a universal algebra such that $S \cap T \neq \emptyset$ for all Rees subalgebras $S$, $T$ of $\mathfrak{A}$. If $\mathfrak{A}$ does not contain an idempotent element, then $(R(\mathfrak{A}), \wedge, \vee)$ is isomorphic to a sublattice of the congruence lattice $\mathcal{S}(\mathfrak{A})$ of $\mathfrak{A}$. Generally this does not hold (see the semilattice $S_0$).*

| $S_0$ | 0 | $e$ | $f$ | $p$ | $q$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| $e$ | 0 | $e$ | $e$ | 0 | 0 |
| $f$ | 0 | $e$ | $f$ | 0 | 0 |
| $p$ | 0 | 0 | 0 | $p$ | $p$ |
| $q$ | 0 | 0 | 0 | $p$ | $q$ |

Proof: Because of Proposition 2.4. the union $S \cup T$ of two Rees subalgebras $S$, $T$ is a Rees subalgebra again. So we have $S \vee T = S \cup T$ for all Rees subalgebras $S$, $T$ of $\mathfrak{A}$.

We define the mappipg $\varphi\colon R(\mathfrak{A}) \to \mathcal{S}(\mathfrak{A})$ by $\varphi(S) = \rho_S$ if $S$ is a Rees subalgebra and $\varphi(\emptyset) = id$. Since $\mathfrak{A}$ does not contain an idempotent element, $\varphi$ is injective; we obtain

$$\varphi(S \wedge T) = \varphi(S \cap T) = \rho_{(S \cap T)} = \rho_S \wedge \rho_T = \varphi(S) \wedge \varphi(T)$$
$$\varphi(S \vee T) = \varphi(S \cup T) = \rho_{S \cup T} = \rho_S \vee \rho_T = \varphi(S) \vee \varphi(T);$$

so $\varphi$ is a lattice homomorphism; hence $\varphi(R(\mathfrak{A}))$ is a sublattice of $\mathcal{S}(\mathfrak{A})$.

## 3. Applications to polynomial algebras

In this chapter we consider for an algebra $\mathfrak{A} = (A, \Omega)$ the algebra $\mathcal{P}(\mathfrak{A}, n)$ of all polynomial functions in $n$ variables over $\mathfrak{A}$, the operations of which are the operations of $\mathfrak{A}$ pointwise and the $n+1$-ary composition $o_n$. If $p_i$ $(i = 1, \dots, n+1)$ are $n+1$ polynomial functions of $\mathcal{P}(\mathfrak{A}, n)$ represented by the polynomials $p_i(x_1, \dots, x_n)$, then the composition $o_n(p_1, \dots, p_{n+1})$ of the polynomial functions $p_i$ is just the polynomial function represented by the polynomial

$$p_i(p_2(x_1, \dots, x_n), \dots, p_{n+1}(x_1, \dots, x_n)),$$

where the variable $x_i$ occuring in $p_1$ is substituted with $p_{i+1}(x_1, \dots, x_n)$.

Proposition 3.1. *Let $\mathfrak{A} = (A, \Omega)$ be an idempotent algebra such that for every polynomial function $f: A \to A$ $f \circ f = f$. Then $\mathcal{P}(\mathfrak{A}, n)$ is an idempotent algebra.*

Proof: We proceed by induction; for $n = 1$ it is clear because of the condition of the proposition. We have to show that $\mathcal{P}(\mathfrak{A}, n+1)$ is idempotent, if $\mathcal{P}(\mathfrak{A}, n)$ is idempotent; the only operation we must consider is $o_n$. Let $p = p(x_1, \dots, x_{n+1})$ be such a polynomial function; if one variable does not occur in the word representation of this function the proposition is clear by the assumption of the induction. Let $\mathbf{x}$ be the vector $(x_1, \dots, x_{n+1})$. We use the condition $f \circ f = f$ for all $f$ defined by $f(x_{n+2}) = p(\mathbf{x}), x_1, \dots, x_n$ fixed; therefore

$$\text{(i)} \qquad p(x_1, \dots, x_n, p(\mathbf{x})) = p(\mathbf{x}).$$

If we define $g(x_1, \dots, x_n) = p(\mathbf{x})$ for fixed $x_{n+1}$, we obtain

$$g(g(x_1, \dots, x_n), \dots, g(x_1, \dots, x_n)) = g(x_1, \dots, x_n);$$

and so

$$\text{(ii)} \qquad p(p(\mathbf{x}), \dots, p(\mathbf{x}), x_{n+1}) = p(\mathbf{x}).$$

Put $y = (p(\mathbf{x}), \dots, p(\mathbf{x}), x_{n+1})$ and combining (i) and (ii) we obtain

$$p(p(\mathbf{x}), \dots, p(\mathbf{x})) = p(p(\mathbf{x}), \dots, p(\mathbf{x}), p(y)) = p(y) = p(\mathbf{x});$$

Corollary 3.2. *Let $\mathfrak{A} = (A, \Omega)$ be an idempotent algebra such that $f \circ f = f$ for all polynomial functions $f: A \to A$. If $\rho$ is a congruence on $\mathcal{P}(\mathfrak{A}, n)$ with at most one non-singleton class, then $\rho$ is a Rees congruence.*

This is obvious; because $\mathcal{P}(\mathfrak{A}, n)$ does not contain a nullary operation. One can apply Proposition 2.17.

Remark 3.3. Considering the lattice $(L, \wedge, \vee)$ we obtain the following result:

$L$ is distributive if and only if $\mathcal{P}(L,n)$ is idempotent. For $n = 1$ this is a wellknown theorem of Skhweigert (1975); for $n \in N$ it results, applying Proposition 3.1. So in the case of a distributive lattice $L$ we can apply Corollary 3.2. to the algebra $\mathcal{P}(L,n)$. Trivially all these facts remain true, if we consider the algebra $\mathcal{P}^*(L,n)$, which also consists of all polynomial functions in $n$ variables over $L$, but the only operation is $o_n$.

If $\mathfrak{A} = (S,\cdot)$ is a semilattice, then Proposition 3.1. and Corollary 3.2. are applicable, too.

Proposition 3.4. *Let $\mathfrak{A} = (A;\Omega)$ be a universal algebra and $M$ a Rees subalgebra of $\mathfrak{A}$, such that for every $p \in \mathcal{P}(\mathfrak{A},n) p(x) \in M$, if at least one coordinate of $\mathbf{x} = (x_1,\dots,x_n)$ is an element of $M$. If $I_M = \{p: p(\mathbf{x}) \in M$ for every $\mathbf{x} \in A^n\}$ and $I_M \neq \emptyset$, then $I_M$ is a Rees subalgebra of $\mathcal{P}(\mathfrak{A},n)$.*

Proof: Let $\omega$ be a $k$-ary pointwise operation of $\mathcal{P}(\mathfrak{A},n)$ and $p_1,\dots,p_k \in I_M$. Then $\omega(p_1,\dots,p_k)(\mathbf{x}) = \omega(p_1(\mathbf{x}),\dots,p_k(\mathbf{x})) \in M$ for every $\mathbf{x} \in A^n$ because of the definition of $I_M$. Similary for every $x \in A^n o_n(p_1,\dots,p_n)(\mathbf{x}) \in M$ results. So $I_M$ is a subalgebra of $\mathcal{P}(\mathfrak{A},n)$, if $I_M \neq \emptyset$. Trivially, for every pointwise operation the condition of Definition 1.1. holds. Let $p_1,\dots,p_{n+1} \in \mathcal{P}(\mathfrak{A},n)$, and $p_j \in I_M$, then

$$o_n(p_1,\dots,p_{n+1})(\mathbf{x}) = p_1(p_2(\mathbf{x}),\dots,p_j(\mathbf{x}),\dots,p_{n+1}(\mathbf{x})) \in M$$

for every $x \in A$; so the proposition is proved.

Remark 3.5. Let $(S,\cdot)$ be a semigroup without null and $M$ an idea of $S$. Then $I_M \neq \emptyset$ is a Rees subalgebra of $\mathcal{P}(S,n)$ and defines a Rees congruence $\rho_M$ on $\mathcal{P}(S,n)$. If $N$ is an ideal different from $M$, then $I_M \neq I_N$ because $\mathcal{P}(S,n)$ contains all constant functions. If we define $\rho_\emptyset = id$, the mapping $M \to \rho_M$ is a lattice isomorphism from $\mathcal{J}(S) \cup \{\emptyset\}$ into the congruence $\mathcal{C}(\mathcal{P}(S,n))$ where $\mathcal{J}(S)$ is the ideal lattice of $(S,\cdot)$. So $\mathcal{J}(S) \cup \{\emptyset\}$ is isomorphic to a sublattice of $\mathcal{C}(\mathcal{P}(S,n))$: $S$ must be convergence free if $\mathcal{C}(\mathcal{P}(S,n))$ is congruence free.

Remark 3.6. Let $(L,\wedge,\vee)$ be a lattice with more than two elements; then a non-trivial ideal $M$ exists; which defines $I_M$ as in Proposition 3.4. and $I_M \neq \emptyset$. $I_M$ defines a non-travial Rees congruence on $\mathcal{P}^*(L,n)$; so $\mathcal{P}^*(L,n)$ cannot be congruence free. This result remains true, if we consider the algebra of all those polynomial functions, in the representation of which all variables $x_1,\dots,x_n$ occur.

## REFERENCES

[1] H. Lausch – W. Nölower, *Algebra of Polynomials*, North – Holland Publ. (1973).

[2] D. Rees, *On Semigroups*, Proc. Cambridge Phil. Soc. **36** (1940), 387–400.

[3] D. Schweigert, *Über idempotente Polynomfunktionen auf Vierbänden*, Elem. d. Math. **30** (1975), 30–32.

[4] G. Ssasz, *Rees factor lattices*, Publ. Math. Debrecen **15** (1968), 259–266.

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