

q -ANGELESCU POLYNOMIALS

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1. Introduction: The Angelescu polynomial is defined by

$$\bar{\Lambda}_n(x) = e^x D^n [e^{-x} A_n(x)], \quad (1.1)$$

where $\frac{A_n(x)}{n!}$ is an Appell set of polynomials. In a recent communication I have studied the polynomial

$$\bar{\Lambda}_n^{(a)}(x) = x^{-a} e^x D^n [x^a e^{-x} A_n(x)], \quad (1.2)$$

where a is a constant. In the present paper we shall make a study of a q -Angelescu polynomial which is an extension of the Angelescu polynomial defined by (1.1), q -Appell sets were first defined and studied by Sharma and Chak [2] who called these sets as q -harmonic. Later on, these sets were studied by Al-Salam [1].

We define q -Angelescu polynomials as

$$\bar{\Lambda}_{n,q}(x) = e_q(xq^n) D_q^n [E_q(-x) P_n(x)], \quad (1.3)$$

where $P_n(x)$ is a q -Appell set satisfying the property

$$D_q \{P_n(x)\} = [n] P_{n-1}(x)$$

and

$$D_q \{f(x)\} = \frac{f(xq) - f(x)}{x(q-1)}.$$

2. Preliminaries: Let α be real or complex and let

$$[\alpha] = (1 - q^\alpha).$$

We shall use the notation

$$(a; k) = (1 - q^a) \dots (1 - q^{a+k-1}), \quad (a)_0 = 1,$$

$$\begin{bmatrix} \alpha \\ k \end{bmatrix} = (-1)^k q^{k(2\alpha-k+1)/2} \frac{(-\alpha; k)}{(1; k)}$$

and

$${}_1\theta_1 \left(a; b; x; \frac{1}{2} \right) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} (a; n) x^n}{(1; n)(b; n)}.$$

Let us recall the well-known formula

$$[a + b]_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k} b^k. \tag{2.1}$$

There are two q -analogues of exponential function e^x in common use. They are

$$e_q(x) = \prod_{n=0}^{\infty} (1 - q^n x)^{-1} = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} \tag{2.2}$$

and

$$E_q(x) = \prod_{n=0}^{\infty} (1 + q^n x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} x^k}{[k]!} \tag{2.3}$$

where $[k]! = [1] \dots [k]$.

3. A Generating function for $\bar{\Lambda}_{n,q}(x)$.

By definition

$$\begin{aligned} \bar{\Lambda}_{n,q}(x) &= e_q(xq^n) D_q^n [E_q(-x) P_n(x)] \\ &= e_q(xq^n) \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} D_q^r \{E_q(-x)\} \Big|_{x=xq^{r-n}} D_q^{n-r} \{P_n(x)\} \\ &= e_q(xq^n) \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} (-1)^r q^{r(r-1)/2} E_q(-xq^n) \times \\ &\quad \times P_r(x) [n][n-1] \dots [r+1]. \end{aligned}$$

Hence, after suitable adjustment, we have the generating relation

$$\sum_{n=0}^{\infty} \frac{\bar{\Lambda}_{n,q}(x) t^n}{(1; n)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} t^n P_n(x)}{(1; n)[1-t]_{n+1}}, \tag{3.1}$$

where ${}_1\theta_0(n+1; -; t)$ is written as $\{[1-t]_{n+1}\}^{-1}$.

It does not seem possible to express the right hand expression in simpler terms as in the case of Angelescu polynomials $\bar{\Lambda}_n(x)$.

4. A Characterisation for $\bar{\Lambda}_{n,q}(x)$.

We now proceed to give a characterization for the q -Appell polynomial $\bar{\Lambda}_{n,q}(x)$.

Sharma and Chak [2] gave the following characterization for the q -Appell polynomials: –

A polynomial set $\{P_n(x)\}$ is q -Appell, if and only if, there is a set of constants $\{a_k\}$ such that

$$P_n(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} a_{n-k} x^k, \quad a_0 \neq 0. \tag{4.1}$$

Therefore, from (1.3) and (4.1), we have

$$\begin{aligned} \bar{\Lambda}_{n,q}(x) &= e_q(xq^n) D_q^n \left[E_q(-x) \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} a_{n-k} x^k \right] \\ &= e_q(xq^n) \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} a_{n-k} D_q^n [E_q(-x) x^k] \\ &= e_q(xq^n) (-1)^n q^{n(n-1)/2} \sum_{k=0}^n (-n; k) \begin{bmatrix} n \\ k \end{bmatrix} q^k \times \\ &\quad \times a_{n-k} {}_1\theta_1 \left(n+1; n+1-k; -xq^{n-k}; \frac{1}{2} \right). \end{aligned} \tag{4.2}$$

Using a transformation due to Slater [4], namely,

$${}_1\theta_1(a; c; x) = \prod_{n=0}^{\infty} \{1/(1-xq^n)\} {}_1\theta_1 \left(c-a; c; -xq^a; \frac{1}{2} \right) \tag{4.3}$$

we have, from (4.2) that

$$\bar{\Lambda}_{n,q}(x) = (-1)^n q^{n(n-1)/2} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) a_{n-k} q^k \times {}_1\theta_1(-k; n+1-k; xq^n), \tag{4.4}$$

where ${}_1\theta_1(-k; n+1-k; xq^n)$ is a q -Laguerre polynomial.

Thus, we arrive at the characterization that a polynomial $\bar{\Lambda}_{n,q}(x)$ is q -Angelescu, if and only if, there is a set of constants $\{ak\}$ such that

$$\bar{\Lambda}_{n,q}(x) = (-1)^n q^{n(n-1)/2} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) a_{n-k} q^k {}_1\theta_1(-k; n+1-k; xq^n)$$

with $a_0 \neq 0$.

The form (4.4) will be some times denoted as $\overline{\Lambda}_{n,q}(x; a_n)$.

Particular Cases

(i) Taking $a_k = 1$ in (4.4), we get that

$$\overline{\Lambda}_{n,q}(x; 1) = (-1)^n q^{n(n-1)/2} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) q^k {}_1\theta_1(-k; n+1-k; xq^n) \quad (4.5)$$

is an q -Angelescu polynomial.

(ii) Next, if we take

$$a_k = (-1)^k q^{k(k+\frac{1}{2})} \text{ in (4.4)}$$

we get the q -Angelescu polynomial

$$\begin{aligned} \overline{\Lambda}_{n,q}\left(x; (-1)^k q^{k(k+\frac{1}{2})}\right) &= (-1)^n q^{3n^2/2} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) \times \\ &\quad \times q^{k(k+\frac{1}{2})-2nk} {}_1\theta_1(-k; n-k+1; xq^n). \end{aligned}$$

This can be alternatively written as

$$\begin{aligned} \overline{\Lambda}_{n,q}\left(x; (-1)^k q^{k(k+\frac{1}{2})}\right) &= e_q(xq^n) q^{3n^2/2} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) \times \\ &\quad \times q^{k(k+\frac{1}{2})-2nk} {}_1\theta_1\left(n+1; n+1-k; -xq^{n-k}; \frac{1}{2}\right). \end{aligned}$$

5. Another characterization of q -Angelescy polynomial

We shall now give a characterization of $\overline{\Lambda}_{n,q}(x)$ in terms of moment constants.

Al-Salam [1] has proved the following characterization of q -Appell polynomial:

“A polynomial set $\{P_n(x)\}$ is a q -Appell set, if and only if, there is a function $\beta(x; q) \equiv \beta(x)$ of bounded variation on $(0, \infty)$ so that

$$b_n = \int_0^\infty x^n d\beta(x), \text{ exists for all } n = 0, 1, 2, \dots, b_0 \neq 0, \quad (5.1)$$

$$P_n(x) = \int_0^\infty [x+t]_n d\beta(t). \quad (5.2)$$

Using (5.2), we have

$$\begin{aligned} \bar{\Lambda}_{n,q}(x) &= \exists_q(xq^n)D_q^n \left[E_q(-x) \int_0^\infty [x+t]_n d\beta(t) \right] \\ &= e_q(xq^n) \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \int_0^\infty t^k d\beta(t) D_q^n [E_q(-x)x^{n-k}] \\ &= e_q(xq^n) \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} b_k D_q^n [E_q(-x)x^{n-k}] \\ &= (1; n)e_q(xq^n) \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{b_k q^{k(k-1)/2}}{(1; k)} \times \\ &\quad \times {}_1\theta_1 \left(n+1; k+1; -xq^k; \frac{1}{2} \right). \end{aligned}$$

The above expression for $\bar{\Lambda}_{n,q}(x)$ can also be written in the alternative form [using 4.3]

$$\bar{\Lambda}_{n,q}(x) = (1; n) \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{b_k q^{k(k-1)/2}}{(1; k)} {}_1\theta_1(k-n; k+1; xq^n).$$

Thus, we have the characterization that a polynomial set $\bar{\Lambda}_{n,q}(x)$ is a q -Angelescu set, if and only if, there exist constants b_k such that there is a function $\beta(x)$ of bounded variation on $(0, \infty)$ so that

$$b_n = \int_0^\infty x^n d\beta(x), \quad n \geq 0, \quad b_0 \neq 0,$$

with

$$\bar{\Lambda}_{n,q}(x) = (1; n) \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \frac{b_k q^{k(k-1)/2}}{(1; k)} {}_1\theta_1(k-n; k+1; xq^n).$$

6. Transformation relations of certain particular q -Angelesca polynomial

Consider the polynomials (4.4)

$$\bar{\Lambda}_{n,q}(x; 1) = (-1)^n q^{n(n-1)/2} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) {}_1\theta_1(k-n; n+1-k; xq^n) \quad (6.1)$$

Using the following known transformation due to Slater [4], namely

$${}_1\theta_1(a; b+1; x) = \frac{(1-q^b)}{x} \{ {}_1\theta_1(a; b; x) - {}_1\theta_1(a-1; b; xq^{1-a}) \}$$

on the right hand side of (6.1), we get

$$\begin{aligned} \bar{\Lambda}_{n,q}(x; 1) &= \frac{(1-q^n)(q^n-1)}{qx} \bar{\Lambda}_{n-1,q}(xq; a'_{n-1}) - \frac{(-1)^n q^{n(n-1)/2}}{x} \times \\ &\quad \times \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) (1-q^{n-k}) q^k {}_1\theta_1(-k-1; n-k; xq^{n+k+1}), \end{aligned}$$

where $a'_{n-1} = \frac{q^n}{(1-q^n)}$.

Similarly from (6.1) and using Slater's [4; 2.1, 2.2, 2.6] transformations we get

$$\begin{aligned} \bar{\Lambda}_{n,q}(x; 1) &= \frac{(-1)^n q^{n(n+1)/2}}{(q-q^{n+1})} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) q^k (1-q^{1-k}) \times \\ &\quad \times {}_1\theta_1(2-k; n-k+1; xq^k) + \\ &\quad + \frac{(-1)^n q^{n(n+1)/2} (x-1)}{(q^n-1)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) {}_1\theta_1(1-k; n-k+1; xq^n) + \\ &\quad + \frac{(q^n+q^{n-1}-1)(-1)^n q^{n(n-1)/2}}{(q^n-1)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) q^k {}_1\theta_1(1-k; n-k+1; xq^n), \\ \bar{\Lambda}_{n,q}(x; 1) &= (-1)^n q^{n(n-1)/2} (q^{n+3}-q) \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) \frac{q^k}{(q^{n+1-k}-1)} \times \\ &\quad \times {}_1\theta_1(-k-2; n-k+1; xq^n) + \\ &\quad + (-1)^n q^{(n-1)/2} q^{n-1} (q^2-x) \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-n; k)}{(q^{n+1-k}-1)} {}_1\theta_1(-k-1; n-k+1; xq^n) + \\ &\quad + (-1)^n q^{n(n-1)/2} (q-q^{n+2}-q^{n+3}) \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-n; k) q^k}{(q^{n+1-k}-1)} {}_1\theta_1(-k-1; \\ &\quad \quad \quad n-k+1; xq^n) \end{aligned}$$

and

$$\begin{aligned} \bar{\Lambda}_{n,q}(x; 1) &= \frac{(-1)^n q^{n(n-1)/2}}{(q^n-1)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) (q^k-1)(q^k-q^n) \times \\ &\quad \times {}_1\theta_1(-k+1; n-k; xq^n) + \\ &\quad + \frac{(-1)^n q^{n(n-1)/2}}{x(q^n-1)} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} (-n; k) [(q^k-q^n) + (x-1)(1-q^{n-k})] q^k \times \\ &\quad \times {}_1\theta_1(-k; n-k; xq^n) \end{aligned}$$

respectively.

7. An integral involving $\bar{\Lambda}_{n,q}(x)$.

We now proceed to derive the value of a general integral involving $\bar{\Lambda}_{n,q}(x)$. In particular, depending on the orthogonality of the Appell polynomial, assume that q -Appell polynomial $P_n(x)$ is orthogonal in the interval (α, β) with respect to a normalized weight function $w(x)$. Consider then the integral

$$\int_{\alpha}^{\beta} w(x)\bar{\Lambda}_{n,q}(x)\bar{\Lambda}_{m,q}(x)dx = \int_{\alpha}^{\beta} w(x) \sum_{r=0}^r \sum_{s=0}^m \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} m \\ s \end{bmatrix} (-1)^{r+s} q^{r(r-1)/2} \times \quad (7.1)$$

$$\times q^{s(s-1)/2} P_r(x)P_s(x)(r+1;n)(s+1;m)dx$$

Using the assumed orthogonality property for $P_n(x)$, we get

$$\int_{\alpha}^{\beta} w(x)\bar{\Lambda}_{n,q}(x)\bar{\Lambda}_{m,q}(x)dx = \sum_{r=s=0}^{\min(m,n)} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} m \\ s \end{bmatrix} (-1)^{r+s} \times$$

$$\times q^{r(r-1)/2+s(s-1)/2}(r+1;n)(s+1;m)$$

$$= 0, \text{ otherwise.}$$

As for example, consider the q -Appell polynomial $H_n(x)$, defined as

$$H_n(x) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

Since the set $H_n(-xq^{-\frac{1}{2}})$ is orthogonal over the unit circle with respect to the weight function

$$f(\theta) = \sum_{n=-\infty}^{\infty} q^{n^2/2} e^{in\theta}, \quad |q| < 1,$$

we have from (7.1), using orthogonality condition for $H_n(-zq^{-\frac{1}{2}})$

$$\int_{|x|=1} f(x)\bar{\Lambda}_{n,q}\left(-xq^{-\frac{1}{2}}; 1\right)\bar{\Lambda}_{m,q}\left(-xq^{-\frac{1}{2}}; 1\right) dx$$

$$= K \sum_{r=s=0}^{\min(n,m)} \begin{bmatrix} n \\ r \end{bmatrix} \begin{bmatrix} m \\ s \end{bmatrix} (-1)^{r+s} q^{r(r-1)/2+s(s-1)/2}(r+1;n)(s+1;m)$$

= 0, otherwise.

where $K = 1$ if $f(x)$ is a normalized weight function, otherwise K is a suitable constant.

8. Still another integral involving an orthogonal q -Appell polynomial and $\bar{\Lambda}_{n,q}(x)$.

Again considering a set of q -Appell polynomials $P_n(x)$ orthogonal with respect to a normalized weight function $w(x)$ over the interval (α, β) we have

$$\int_{\alpha}^{\beta} w(x) P_m(x) \bar{\Lambda}_{n,q}(x) dx$$

$$= \int_{\alpha}^{\beta} w(x) P_m(x) \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} (-1)^r q^{r(r-1)/2} (r+1; n) P_r(x) dx.$$

Again using the orthogonality property for $P_n(x)$, we have

$$\int_{\alpha}^{\beta} w(x) P_m(x) \bar{\Lambda}_{n,q}(x) dx = 0; \text{ if } m > n$$

$$= (-1)^n q^{n(n-1)/2} (n+1; n); \text{ } m = n$$

$$= (-1)^m q^{m(m-1)/2} (m+1; n) \text{ } m < n.$$

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REFERENCES

- [1] Al-Salam W. A., *q-Appell polynomials*. Annali di Mathematica pura et applicata (iv) Vol. LXXVII 31-46.
- [2] Sharma, A. and Chak, A., *The basic analogue of a class of polynomials*. Rivista di Matematica della universita di Parma Vol. 5, 1954, 325-337.
- [3] Shukla, D. P., *Generalised Angelescu polynomial*. Communicated for publication.
- [4] Slater, L. J., *The evaluation of the basic confluent hypergeometric function*. Proc, Cam. Phil. Soc., 50, (30), 1854, 404-13.

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