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APPLICATION OF CHAPLIGIN'S THEOREM TO AUTONOMOUS SYSTEMS[1]

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The aim of this paper is to apply the Chapligin's method for constructing boundary curves to some first order differential equations. For these equations other authors observed the topological structure of integral curves or the existence and the number of limit cycles.

Since in the Poincaré-Bendixson theory, which deals with these problems, there occur systems

$$\frac{dx}{dt} = M(x,y), \qquad \frac{dy}{dt} = N(x,y) \tag{1}$$

(autonomous systems) and having in mind that solutions of (1) are paths $x = g(t)$, $y = h(t)$ in a phase plane, we may ask if the Chaplygin's method might be extended to be valid not only for solutions of the differential equation but even for solutions of the system (1). We shall state a theorem for an autonomous system which is an analogue to the Chapligin's theorem for a first order differential equation.

THEOREM. *Consider the system* (1), *where* $M(x,y)$ *and* $N(x,y)$ *are continuous functions defined on some domain* D *and satisfy Lipschitz condition with respect to* x *and* y *in every closed domain of* D, *and in that domain let* $M(x,y) > 0$. *Let* $x = g(t)$, $y = h(t)$ *be a unique solution of the system* (1), *satisfying the initial condition* $g_0(t_0) = x_0$, $h(t_0) = y_0$, *which defines a path* $\gamma(t)$ *in a phase plane. Let* $u(x)$ *and* $v(x)$ *be functons such that* $u(x_0) = v(x_0) = y_0$ *and let*

$$\begin{aligned} &M(x,u)u'(x) < N(x,u),\\ &M(x,v)v'(x) > N(x,v) \end{aligned} \tag{2}$$

[1]Presented at Colloquium on Oualitative Theory of Differential Equations, Szeged (Hungary), 30^{th} of August, 1979. The paper is a part of my Doctoral dissertation, which was led by prof. M. Bertolino.

in the domain D for $x \geq x_0$. Then the path $\gamma(t)$ does not intersect the curves defined by $x = t$, $y = u(t)$ and $x = t$, $y = v(t)$ for $t > t_0$.

PROOF. – According to the hypothesis, the function $\frac{N(x,y)}{M(x,y)}$ is continuous in D and satisfies Lipschitz condition in every closed domain in D, since from the inequalities

$$|M(x_2, y_2) - M(x_1, y_1)| < k_1(|x_2 - x_1| + |y_2 - y_1|),$$
$$|N(x_2, y_2) - N(x_1, y_1)| < k_2(|x_2 - x_1| + |y_2 - y_1|)$$

we obtain

$$\left|\frac{N(x, y_2)}{M(x, y_2)} - \frac{N(x, y_1)}{M(x, y_1)}\right| <$$
$$< \frac{|M(x, y_1)||N(x, y_2) - N(x, y_1)| + |N(x, y_1)||M(x, y_2) - M(x, y_1)|}{|M(x, y_2)||M(x, y_1)|} <$$
$$< \frac{M_1 k_2 |y_2 - y_1| + M_2 k_1 |y_2 - y_1|}{m^2} = k|y_2 - y_1|,$$

where M_1, M_2 and m are constans such that

$$m \leq |M(x, y)| \leq M_1, \qquad |N(x, y) \leq M_2$$

in every closed domain in D. Thus there exists a unique solution $y(x)$ of the differential equation

$$y' = \frac{N(x, y)}{M(x, y)}$$

such that $y(x_0) = y_0$. From the positivity of the derivative $\frac{dx}{dt}$ we see that $x > x_0$ for $t > t_0$. Consequently, according to the Chapligin's theorem for a first order differential equation, there follows

$$u(x) < y(x) < v(x).$$

This completes the proof.

REMARK 1. – If $M(x, y) < 0$, the formulation of the theorem is analogous, but $\gamma(t)$ does not intersect the corresponding curves for $t < t_0$.

REMARK 2. – If the inequalities (2) hold for $x \leq x_0$, $\gamma(t)$ does not intersect the corresponding curves for $t < t_0$ if $M(x, y) > 0$, and for $t > t_0$ if $M(x, y) < 0$.

REMARK 3. –If the function $N(x, y)$ is constant sign (let $N(x, y) > 0$), the formulation of the theorem is analogous; instead of functions $u(x)$ and $v(x)$, one considers functions $u(y)$ and $v(y)$, the inequalities (2) change to

$$N(u, y)u'(y) < M(u, y), \quad N(v, y)u'(y) > M(v, y),$$

and the corresponding curves are defined by $x = u(t)$, $y = t$ and $x = v(t)$, $y = t$.

The question is whether the assumption $M(x,y) > 0$, which leads to the existence of a function as a solution of an autonomous system instead of any path in a phase plane, is necessary or not. In order to show the necessity of this assumption, consider the system

$$\frac{dx}{dt} = x - y, \qquad \frac{dy}{dt} = x + y.$$

One the solutions of this system, whose path passes trought the point $(1,0)$ as $t = 0$, is $x = e^t$ cost, $y = e^t$ sint. The function $u(x) = 0$ satisfies the inequality

$$(x - 0) \cdot 0 < x + 0$$

for $x \geq 1$ and passes trough $(1,0)$ but still the corresponding curve and the integral curve intersect in every point $(e^{2k\pi}, 0)$ for $k = 1, 2, \ldots$

Thus, we have only an anlogue to the Chapligin's theorem for a first order differential equation and a proper extension to an autonomous system cannot be made.

1. Observe the differential equation

$$y' = -\frac{x(ax + by + c)}{y(ax + by + c) + x^2 + y^2 - 1}. \tag{3}$$

The boundary curves of the solution $y(x)$ passing trough the point (x_0, y_0) in the domain

$$\Omega = \{(x,y) \colon x \neq o, (ax + by + c)(x^2 + y^2 - 1)y > 0\},$$

where $(x_0, y_0) \in \Omega$, are the straight line $y = y_0$ and the circle $x^2 + y^2 = x_0^2 + y_0^2$.

To prove this, notice that if $ax + by + c > 0$, $x^2 + y^2 - 1 > 0$, $y > 0$, $x > 0$, there holds

$$-\frac{x}{y} - \frac{x(ax + by + c)}{y(ax + by + c) + x^2 + y^2 - 1} < 0,$$

and, according to the solutions of the equations $y' = -x/y$ and $y' = 0$ we find that for $x > x_0$

$$\sqrt{x_0^2 + y_0^2 - x^2} < y(x) < y_0.$$

One obtains the same inequality for $ax + by + c < 0$, $x^2 + y^2 - 1 < 0$, $y > 0$, $x > 0$.

In the same manner we obtain that for $ax + by + c > 0$, $x^2 + y^2 - 1 < 0$, $y < 0$, $x > 0$, as well as for $ax + by + c < 0$, $x^2 + y^2 - 1 > 0$, $y < 0$, $x > 0$

$$y_0 < y(x) < -\sqrt{x_0^2 + y_0^2 - x^2}.$$

If $ax + by + c > 0$, $x^2 + y^2 - 1 > 0$, $y > 0$, $x < 0$ or $ax + by + c < 0$, $x^2 + y^2 - 1 < 0$, $y > 0$, $x < 0$, there follows

$$y_0 < y(x) < \sqrt{x_0^2 + y_0^2 - x^2}.$$

For $ax+by+c>0$, $x^2+y^2-1<0$, $y<0$, $x<0$ and for $ax+by+c<0$, $x^2+y^2-1>0$, $y<0$, $x<0$ we have

$$-\sqrt{x_0^2+y_0^2-x^2}<y(x)<y_0.$$

Qin Yuan-xun |2| proved that a necessary and sufficient condition for a quadratic system to have algeabraic limit cycles of second degree is that it may be transformed by a nonsingular linear transformation into the form (3), where constants a, b and c satisfy conditions $c^2>a^2+b^2$, $a\neq 0$. He also proved the uniqueness and stability and presented practical computational procedures of such limit cycles. These conclusions follow from the characteristics of the equations (3). Obviously, the unit circle $x^2+y^2=1$ is one of the solutions, which is, due to $c^2>a^2+b^2$, periodic. If $a=0$, one obtains that the critical point inside the unit circle is a center. If $a\neq 0$ the critical point is a focus (precisely, the author proves that for $b\neq 1$ the critical point inside the circle is a focus or a node; a later result of Vorobyew in 1960, that a critical point inside a closed curve is not a node defines this). A qualitative picture of the paths is studied first for $a=0$ and then for $a\neq 0$. Consequently, the author proves that for $b=-1$ there is only one critical point i.e. a center for $a=0$ and a focus for $a\neq 0$, for $b<-1$ there are two centers (two focal points), and for $b>-1$ a center (a focus) and a saddle.

2. The equation

$$y'=\frac{-x+bxy+cy^2+ay}{y+y^2} \tag{4}$$

has been studied in several papers. Ye Yan-qian, He Chang-you, Wang Ming-shu, Xu Ming-wei and Luo Ding-jun proved in 1963. that for $c=0$ there are no limit cycles. Averin proved in 1966. that one can find numbers $a>0$ and $b>2$ such that there are two limit cycles enclosing the origin. Cherkas and Zilevich |7|, |8|, |9| proved for $a\geq 0$, $c\geq 0$ the existence of a most one limit cycle enclosing $(0,0)$.

Here we assume that $a\geq 0$, $c\geq 0$. Suppose $b\geq 0$; then we find that a solution $y(x)$ with an initial condition $y(x_0)=y_0$, in the domain $x>0$, $y>0$ satisfies for $x>x_0$ the inequalities

$$\sqrt{x_0^2+y_0^2-x^2}<y(x)<\left(\frac{b}{c}x_0+y_0+\frac{b+ac}{c^2}\right)e^{c(x-x_0)}-\frac{b}{c}x_0-\frac{b+ac}{c^2}\qquad (c>0),$$

$$\sqrt{x_0^2+y_0^2-x^2}<y(x)<\frac{b}{c}(x^2-x_0^2)+a(x-x_0)\qquad (c=0).$$

These inequalities follow from

$$-\frac{x}{y}<\frac{-x+bxy+cy^2+ay}{y+y^2}<bx+cy+a$$

and from the solutions of the corresponding differential satisfying the initial condition $y(x_0)=y_0$.

Suppose $b < 0$; in the same domain there hold the inequalities

$$q^{-1}(x) < y(x) < \left(y_0 + \frac{a}{c}\right) e^{c(x-x_0)} - \frac{a}{c} \qquad (c > 0),$$
$$q^{-1}(x) < y(x) < a(x - x_0) + y_0 \qquad (c = 0),$$

where $q^{-1}(x)$ is the inverse function of the function

$$q(x) = \sqrt{x_0^2 + \frac{2}{b}(x - y_0) + \frac{2}{b^2} \ln \frac{1 - bx}{1 - by_0}}.$$

This follows from

$$\frac{by - 1}{y} x < \frac{-x + bxy + cy^2 + ay}{y + y^2} < cy + a$$

and one obtains the above inequalites by intergrating the corresponding equations.

A possibility of the existence of a limit cycle follows also from our results. The lower boundary curve is a circle for $b \geq 0$; for $b < 0$ is given by

$$x = \sqrt{x_0^2 + \frac{2}{b}(y - y_0) + \frac{2}{b^2} \ln \frac{1 - by}{1 - by_0}}$$

and this curve intersect x axis as $x = \sqrt{x_0^2 + \frac{2}{b^2}\{-by_0 - \ln(1 - by_0)\}} > x_0$. Thus the integral curve passing through (x_0, y_0) does not necessarily remain in a domain bounded by boundary curves.

3. For an equation

$$y' = \frac{a_0 x^n + a_1 x^{n-1} y + \ldots + a_n y^n}{b_0 x^n + b_1 x^{n-1} y + \ldots + b_n y^n}, \tag{5}$$

where

$$\sum_{i=0}^{n} a_i^2 > 0, \quad \sum_{i=0}^{n} b_i^2 > 0,$$

and a_i or $b_i (i = 0, 1, \ldots, n)$ are of the same sign or egual to zero, the boundary curves passing through (x_0, y_0) in the domain

$$\Omega = \{(x, y) \colon y \neq x, xy > 0\}$$

are

$$|y| = \left| \sqrt[n+1]{M(x^{n+1} - x_0^{n+1}) + y_0^{n+1}} \right|$$

and

$$|y| = \left| \sqrt[n-1]{M\left(\frac{1}{x^{n-1}} - \frac{1}{x_0^{n-1}}\right) + \frac{1}{y_0^{n-1}}} \right|^{-1},$$

where

$$M = \frac{a_0 + a_1 + \ldots + a_n}{b_0 + b_1 + \ldots + b_n}.$$

More precisely, under the assumptions that $a_i (i = 0, 1, \ldots, n)$ are of the same sign as $b_i (i = 0, 1, \ldots, n)$ and $y > x > 0$, or that $a_i (i = 0, 1, \ldots, n)$ are of opposite sign reffered to $b_i (i = 0, 1, \ldots, n)$ and $x > y > 0$, there holds for $x > x_0$

$$\sqrt[n+1]{M(x^{n+1} - x_0^{n+1}) + y_0^{n+1}} < y(x) < \frac{1}{\sqrt[n-1]{M\left(\frac{1}{x^{n-1}} - \frac{1}{x_0^{n-1}}\right) + \frac{1}{y_0^{n-1}}}}.$$

If a_i are of the same sign as b_i and $x > y > 0$, or a_i are of opposite sign refered to b_i and $y > x > 0$, one gets for $x > x_0$

$$\frac{1}{\sqrt[n-1]{M\left(\frac{1}{x^{n-1}} - \frac{1}{x_0^{n-1}}\right) + \frac{1}{y_0^{n-1}}}} < y(x) < \sqrt[n+1]{M(x^{n+1} - x_0^{n+1}) + y_0^{n+1}}.$$

If a_i are of the same sign as b_i and $y < x < 0$, or a_i are of opposite sign refered to b_i and $x < y < 0$, one obtains

$$-\sqrt[n+1]{M(x^{n+1} - x_0^{n+1}) + y_0^{n+1}} < y(x) < \frac{1}{\sqrt[n-1]{M\left(\frac{1}{x^{n-1}} - \frac{1}{x_0^{n-1}}\right) + \frac{1}{y_0^{n-1}}}}$$

for $x > x_0$.

If a_i are of the same sign as b_i and $x < y < 0$, or a_i are of opposite sign refered to b_i and $y < x < 0$, there holds for $x > x_0$

$$-\frac{1}{\sqrt[n-1]{M\left(\frac{1}{x^{n-1}} - \frac{1}{x_0^{n-1}}\right) + \frac{1}{y_0^{n-1}}}} < y(x) < -\sqrt[n+1]{M(x^{n+1} - x_0^{n+1}) + y_0^{n+1}}.$$

Lee Shen-ling |4| calculated thhe number of topological classes of dispositions of paths in a phase plane foe the equation (5), by constructing some special numerical functions. Chhang Die |3| observed the equation (5) for $n = 3$; under various assumtions on the coefficients of the equation he obtained the number of invariant half-line paths, and in that manner the topological structure of paths has been divided into fifteen classes.

4. Consider the equation

$$y' = \frac{\mu x + 2ay + f(x,y)}{y} \qquad (\mu \neq 0). \tag{6}$$

Gukyamuhow |12| assumed that $f(x,y)$ is an analytic function about $(0,0)$ such that in expansion in power series the constant and first order terms are missing. He also assumed that the roots of the characcteristic equation are real (of the same sigh, of opposite sign and double, i.e. respectively $\mu > 0$, $-a^2 < \mu < 0$, $\mu = -a^2$) and studied the nature of integral curves near the origin.

Here we do not necessarily need that $f(x,y)$ is analitic; our assumptions deal only with certain types of boundary conditions.

Let $\mu < 0$ and

$$\Omega = \left\{(x,y)\colon y > 0, \qquad -2a < \frac{f(x,y)}{y} < M\right\},$$

where M is any real constant. Then for the solution $y(x)$ of the equation (6) passing through (x_0, y_0) in the domain Ω for $x > x_0$ we have

$$\sqrt{\mu(x^2 - x_0^2) + y_0^2} < y(x) < (2a + M)(x - x_0) + y_0.$$

If $\mu > 0$ and

$$\Omega = \left\{(x,y)\colon y > k > 0, \qquad -2a < \frac{f(x,y)}{y} < M\right\},$$

in Ω for $x > x_0$ we find

$$\sqrt{\mu(x^2 - x_0^2) + y_0^2} < y(x) < \frac{M}{2k}(x^2 - x_0^2) + (2a + M)(x - x_0) + y_0.$$

Letting $\mu > 0$ and

$$\Omega = \{(x,y)\colon x > 0,\ \ y > 0,\ \ -2ay < f(x,y) < -\mu x\}$$

for $x > x_0$ in Ω we obtain

$$\sqrt{\mu(x^2 - x_0^2) + y_0^2} < y(x) < 2a(x - x_0) + y_0.$$

The error of this approximation in the interval $x_0 < x < x_0 + \alpha$ is

$$R(x) < 2a(x - x_0) + y_0 - \sqrt{\mu(x^2 - x_0^2) + y_0^2} < 2a\alpha.$$

On the other hand, letting $\mu < 0$ and

$$\Omega = \{(x,y)\colon x > 0,\ \ y > 0,\ \ -\mu x < f(x,y) < -2ay\}$$

for $x > x_0$ in Ω we get

$$2a(x - x_0) + y_0 < y(x) < \sqrt{\mu(x^2 - x_0^2) + y_0^2},$$

and hence for the error in $x_0 < x < x_0 + \alpha$ we have

$$R(x)\sqrt{\mu(x^2 - x_0^2) + y_0^2} - 2a(x - x_0) - y_0 < 2a\alpha.$$

5. Consider the equation

$$y' = \frac{ax^2 - x + bxy + cy + dy^2}{y(x+1)}, \tag{7}$$

where $0 < c + d < \alpha$, $b \geq 1$, $0 \leq a \leq \alpha^2(1 - d) - \alpha b$ (α is any positive number). For the solution $y(x)$ such that $y(0) = 1$ there holds for $x > 0$

$$1 < y(x) < \alpha x + 1.$$

To prove this, notice the function $u(x) = 1$ satisfies the inequality

$$u' \frac{ax^2 - x + bxy + cu + du^2}{u(x+1)} = -\frac{ax^2 + (b-1)x + c + d}{x+1} < 0$$

for $x \geq 0$, and $v(x) = \alpha x + 1$ satisfies for $x \geq 0$

$$v' - \frac{ax^2 - x + bxv + cv + dv^2}{v(x+1)} =$$
$$= \frac{(\alpha^2 - a - \alpha b - \alpha^2 d)x^2 + (\alpha^2 + \alpha + 1 - b - \alpha c - 2\alpha d)x + \alpha - c - d}{(\alpha x + 1)(x+1)} > 0.$$

The last inequality follows from

$$\alpha^2 - a - \alpha b - \alpha^2 d \geq 0,$$
$$\alpha - c - d > 0,$$

and by multiplying the first inequality by $\frac{1}{\alpha}$ and the second one by α and adding them we get

$$\alpha^2 + \alpha + 1 - b - \alpha c - 2\alpha d > 1 + \frac{a}{\alpha} > 0.$$

The equation (7) has been studied by Cherkas, Zilevich and Rychkow |5|, |7|, |9|, |10|. They proved that the equation (7) has at most one limit cycle. Due to our result, if there is *a* limit cycle, then the point $(0,1)$ lies outside it.

6. Consider the equation

$$y' = \frac{-x+\lambda y + ax^2 + bxy + cy^2}{y(1+\mu x)}, \tag{8}$$

where $\mu \geq 0$, $0 < \lambda + c < \alpha$, $b \geq 1$, $0 \leq a \leq \alpha^2(\mu - c) - \alpha b(\alpha$ is any positive number). For the solution $y(x)$ such that $y(0) = 1$ there holds for $x > 0$

$$1 < y(x) < \alpha x + 1.$$

To prove this, notice that the function $u(x) = 1$ satisfies for $x \geq 0$ the inequality

$$u' - \frac{-x+\lambda u + ax^2 + bxu + cu^2}{u(1+\mu x)} = \frac{ax^2 + (b-1)x + \lambda + c}{1+\mu x} < 0,$$

and in the same interval for the function $v(x) = \alpha x + 1$ we have

$$v' - \frac{-x+\lambda v + ax^2 + bxv + cv^2}{v(1+\mu x)} =$$

$$= \frac{(\alpha^2\mu - a - \alpha b - \alpha^2 c)x^2 + (\alpha^2 + \alpha\mu - \alpha\lambda + 1 - b - 2\alpha c)x + \alpha - \lambda - c}{(\alpha x + 1)(1+\mu x)} > 0$$

due to

$$\alpha^2 + \alpha\mu - \alpha\lambda + 1 - b - 2\alpha c \geq \alpha^2 + \alpha\mu - \alpha\lambda - \alpha c + 1 - \alpha\mu > 1.$$

Kukles and Rozet |11| obtained some necessary and sufficient conditions for the nonlocal generation of a limit cycle from singular separatix cyccles for the equation (8).

7. Consider the equation

$$y' = \frac{a + bx + cy + dx^2 + exy + fy^2}{1+xy}, \tag{9}$$

where $0 < a < \alpha$, $0 \leq b \leq -\alpha c$, $d \geq 0$, $e \leq a$, $0 < f \leq 1 - \frac{e}{\alpha} - \frac{d}{\alpha^2}$. For the solution $y(x)$ such that $y(0) = 0$ in the domain $x > 0$, $y \geq 0$ one gets

$$0 < h_1(x) < y(x) < h_2(x) < \alpha x,$$

where $h_1(x)$ and $h_2(x)$ are respectively the solutions of the differential equations

$$y' = \left\{-dx^3 - bx^2 + (e-a)x + c\right\} y + dx^2 + bx + a,$$
$$y' = \frac{-dx^3 - bx^2 + (\alpha f + e - a)x + c}{1+\alpha x^2} y + dx^2 + bx + a$$

such that $h_1(0) = h_2(0) = 0$.

The function $u(x) = 0$ satisfies for $x \geq 0$ the inequality

$$u' - \frac{a + bx + cu + dx^2 + exu + fu^2}{1 + xu} = -(dx^2 + bx + a) < 0.$$

and the function $v(x) = \alpha x$ satisfies for $x > 0$ the inequality

$$v' - \frac{a + bx + cv + dx^2 + exv + fv^2}{1 + xv} =$$
$$= \frac{\alpha^2 - d - e\alpha - f\alpha^2)x^2 + (-b - c\alpha)x + \alpha - a}{1+\alpha x^2} > 0.$$

To find the second pair of aproximate solutions $h_1(x)$, $h_2(x)$ we solve the linear differential equations

$$y' = f'_y(x,0)y + f(x,0),$$
$$y' = \frac{f(x,\alpha x) - f(x,0)}{\alpha x} y + f(x,0),$$

where

$$f(x,y) = \frac{a + bx + cy + dx^2 + exy + fy^2}{1 + xy}.$$

Differentiating with respect to y we find

$$f'_y(x,y) = \frac{-dx^3 - bx^2 + (e-a)y + c + fxy^2 + 2fy}{(1+xy)^2}$$
$$f''_{yy}(x,y) = 2\frac{dx^4 + bx^3 + (a-e)x^2 - cx + f}{(1+xy)^2}.$$

Tanja do ovde ispravljeno Due to the posivity of the second derivative, the second, better approximating, pair of solutions may be obtained.

If $-\alpha < a < 0$, $\alpha c \leq b \leq 0$, $\alpha^2(1-f) + \alpha e \leq d \leq 0$ and $y(0) = 0$, in the interval $0 < x < 1/\sqrt{\alpha}$ there holds

$$-\alpha x < y(x) < 0.$$

This follows from the inequalities

$$u' - \frac{a + bx + cu + dx^2 + exu + fu^2}{1 + xu} = \frac{(\alpha^2 - d + e\alpha - f\alpha^2)x^2 + (c\alpha - b)x - \alpha - a}{1 - \alpha x^2} < 0$$

and

$$v' - \frac{a + bx + cv + dx^2 + exv + fv^2}{1 + xv} = -(dx^2 + bx + a) > 0$$

satisfied by the functions $u(x) = -\alpha x$ and $v(x) = 0$ in the interval $0 \leq x < 1/\sqrt{\alpha}$.

Cherkas and Zilevich |6| proved that the equation (9) has no limit cycle if the polynomial

$$P_4(x) = fx^4 - cx^3 + (a - e)x^2 + bx + d$$

can be written as a product

$$P_4(x) = P_2(x)Q_2(x),$$

where

$$P_2(x) = (1 + 2f)x^2 - cx - e$$

and $Q_2(x)$ is a polynomial of degree most two.

8. Consider the equation

$$y' = \frac{a + bx + cy + dx^2 + exy + fy^2}{x^2 + y}, \tag{10}$$

where $0 < a + c + f < \alpha$, $0 \leq b + e \leq \alpha(\alpha - c - 2f)$, $0 \leq d \leq \alpha(1 - e - \alpha f)$. For the solution $y(x)$ such that $y(0) = 1$ for $x > 0$ there holds

$$1 < y(x) < \alpha x + 1.$$

This follows from the inequalities

$$u' - \frac{a + bx + cu + dx^2 + exu + fu^2}{x^2 + u} = -\frac{dx^2 + (b + e)x + a + c + f}{x^2 + 1} < 0,$$

$$v' - \frac{a + bx + cv + dx^2 + exv + fv^2}{x^2 + v} =$$

$$= \frac{(\alpha - d - \alpha e - \alpha^2 f)x^2 + (\alpha^2 - b - \alpha c - e - 2\alpha f)x + \alpha - a - c - f}{x^2 + \alpha x + 1} > 0$$

satisfied by the functions $u(x) = 1$ and $v(x) = 1 + \alpha x$ for $x \geq 0$.

If $-\alpha < a+c+f < 0$, $\alpha(\alpha+c+2f) \leq b+e \leq 0$, $\alpha(e-\alpha f-1) \leq d \leq 0$ $(0<\alpha<2)$, for the solution $y(x)$ such that $y(0)=1$ we have

$$1-\alpha x < y(x) < 1$$

in the interval $x>0$.

To prove this, notice that for $u(x)=1-\alpha x$ and $v(x)=1$ in the interval $x \geq 0$ there holds

$$u' - \frac{a+bx+cu+dx^2+exu+fu^2}{x^2+u} =$$
$$= \frac{(-\alpha-d+\alpha e-\alpha^2 f)x^2+(\alpha^2-b+\alpha c-e+2\alpha f)x-\alpha-a-c-f}{\left(x-\frac{\alpha}{2}\right)^2+1-\frac{\alpha^2}{4}} < 0,$$
$$v' - \frac{a+bx+cv+dx^2+exu+fu^2}{x^2+v} = -\frac{dx^2+(b+e)x+a+c+f}{x^2+1} > 0.$$

Cherkas and Zilevich |8| proved that the equation (10) has no limit cycle if the polynomial

$$P_4(x) = fx^4 - ex^3 + (d-c)x^2 + bx + a$$

can be written in the form

$$P_4(x) = P_2(x)Q_2(x),$$

where

$$P_2(x) = 2fx^2 - (2+e)x + c$$

and $Q_2(x)$ is polynomial of degree most two.

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$$\frac{dy}{dx} = \frac{\sum_{0<i+j<2} a_{ij}x^i y^j}{\sum_{0<i+j<2} b_{ij}x^i y^j},$$

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$$\frac{dy}{dx} = \frac{ax^3+bx^2y+cxy^2+dy^3}{a_1x^3+b_1x^2y+c_1xy^2+dy^3},$$

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$$y' = \frac{a_0x^n+a_1x^{n-1}y+\cdots+a_ny^n}{b_0x^n+b_1x^{n-1}y+\cdots+b_ny^n},$$

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