

ON SOME SOLUTION OF EQUATION
 $\varphi(x) + \varphi(f(x)) = F(x)$
UNDER THE CONDITION THAT F SATISFIES $F(f^p(x)) = F(x)$ ¹

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I

The results of this paper are connected with solutions $\varphi(x)$ of the functional equation

$$(1) \quad \varphi(x) + \varphi(f(x)) = F(x)$$

where $f(x)$ and $f(x)$ are given functions. Many mathematicians have been concerned with equation (1). Some special cases of this equation have been considered by H. Steinhaus [8]. G. H. Hardy [4], R. Raclis [7] and others. Equations of more general type have been examined by M. Ghermanencu [3] and Kitamura [5].

Some papers of M. Kuczma and M. Barjaktarević are of special interest to this paper. In the paper [6] M. Kuczma proves that under some natural assumptions equation (1) possesses infinitely many solutions which continuous for every x this is not a root of the equation

$$(*) \quad f(x) = x.$$

In the same paper, under assumption that the solution is continuous for $x = x_0$. satisfying equation (*), the author proves the existence of a most such colution. Further, he proves that with addition assumptions such a solution exists and it is given by an explicit formula.

M. Barjaktarević in his paper [1] proves four theorems. Two of them, using different regular methods of summability for the series

$$(2) \quad \frac{1}{2}F(b) + \sum_{v=0}^{\infty} (-1)^v \{F(f^v(x)) - F(b)\},$$

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give sufficient conditions for the existence of a solution of equation (1). The other two theorems give sufficient conditions for existence of a unique continuous solution of (1) in $(a, b]$.

The author of this paper, using the same method, examined some solutions of the equation (1) in the papers [9], [10] and [11]. In [9] and [10], using different regular methods of the summability of the series (2), sufficient i. e. necessary and sufficient conditions for the existence of a solution of the equation (1) are given. Also, sufficient and necessary conditions are given, under which the considered regular methods are translatable from the right on a set

$$(3) \quad \mathcal{U} = \left\{ \{s_n(x)\}_{n=0}^{+\infty} \right\}_{x \in [a, b]},$$

in the sense of the definition of translation given in [2], page 21, where $s_n(x)$ is a partial sum of range n of the series (2).

In the [11] equation (1) is considered under the condition that the function F has a special form for which the series (2), in general, diverges. Also conditions, are given the series (2) to be T -sumable ($T = (a_{kn})$ a regular matrix transformation) to φ so that the following relations are valid

$$\begin{aligned} Ts_n(x) &\rightarrow \varphi(x) \\ \varphi(f(x)) + \varphi(x) &= F(x). \end{aligned}$$

Connected with this, in [11] theorem I is proved.

In this paper we consider a similar cases depending on F as in [11], but with a more general function F .

Theorem I of [11] is related to $p = 1$ (see paragraph II) and the results of this paper are related to the case $p > 1$. There are some specific differences between these two cases.

In theorem I of [11] and throughout this paper it is assumed that

1. $f(x)$ is a continuous strongly monotonic function on $[a, b]$ and that

$$f(a) = a, \quad f(b) = b; \quad f(x) > x, \quad x \in (a, b);$$

2. $f^0(x) = x, f^{y+1}(x) = f(f^y(x)), v \in \{0, \pm 1, \pm 2, \dots\}$.

II

Let $p \in \{2, 3, \dots\}$, p fixed and let the function $F(x)$ have the properties

$$(4) \quad \begin{cases} F(f^p(x)) = F(x) & x \in [a, b] \\ F(f^i(x)) \neq F(x) & x \in [a, b), i = 1, 2, \dots, p-1, p > 1 \\ F(x) = g(x) & x \in [x_0, f(x_0)), x_0 \in (a, b) \end{cases}$$

where $g(x)$ is an arbitrary chosen functions with domain $[x_0, f(x_0))$.

Further, sometimes use the following condition (5) on $F(x)$:

$$(5) \quad \left\{ \begin{array}{l} \text{there are points } x_i \text{ } i \in \{0, 1, \dots, p\}, \text{ in } (a, b) \text{ so that} \\ \Delta = \begin{vmatrix} 1F(x_0) & F(f(x_0)) & F(f^2(x_0)) & \dots & F(f^{p-1}(x_0)) \\ 1F(x_1) & F(f(x_1)) & F(f^2(x_1)) & \dots & F(f^{p-1}(x_1)) \\ 1F(x_2) & F(f(x_2)) & F(f^2(x_2)) & \dots & F(f^{p-1}(x_2)) \\ \dots & \dots & \dots & \dots & \dots \\ 1F(x_p) & F(f(x_p)) & F(f^2(x_p)) & \dots & F(f^{p-1}(x_p)) \end{vmatrix} \neq 0. \end{array} \right.$$

We will be concerned with finding solution $\varphi(x)$ on $[a, b]$ of equation (1) that have form

$$(6) \quad \varphi(x) = a + b_0 F(x) + b_1 F(f(x)) + \dots + b_{p-1}(x) F(f^{p-1}(x)).$$

In connection with this the following theorem is valid.

THEOREM 1. *For the equation (1), in which function $F(x)$ has the properties (4), to have a solution $\varphi(x)$ on $[a, b]$ of the form (6), it is sufficient and, if F also satisfies condition (5), it is necessary that*

$$(7) \quad \left\{ \begin{array}{l} p \text{ is odd number} \\ a = 0, \quad b_i = \frac{1}{2}(-1)^i, \quad i \in \{0, 1, \dots, p-1\} \end{array} \right.$$

i.e. that the function $\varphi(x)$ has the form

$$(8) \quad \varphi(x) = \sum_{i=0}^{p-1} (-1)^i F(f^i(x)).$$

PROOF. To prove that the conditions (7) are sufficient it is enough to substitute (8) into (1) in place of $\varphi(x)$. Let us prove that the conditions (7) are necessary under the assumption that (4) and (5) are satisfied.

Now, according to (6), we get

$$(9) \quad \varphi(x) + \varphi(f(x)) - F(x) = 2a + (b_{p-1} + b_0 - 1)F(x) + \sum_{i=1}^{p-1} (b_i + b_{i-1})F(f^i(x)).$$

For $\varphi(x)$ defined with (6), to be a solution of our equation it is necessary (and sufficient) that the right of (9) is identically equal to zero on $[a, b]$. Setting the right side of (9) equal to $x_i (i = 0, 1, \dots, p)$, we get $p+1$ linear equation

$$(10) \quad \beta_0 F(x_i) + \beta_1 F(f(x_i)) + \dots + \beta_{p-1} F(f^{p-1}(x_i)) + 2a = 0 \quad (i = 0, 1, \dots, p),$$

where for the sake of simplicity, we put

$$(11) \quad \begin{cases} \beta_0 = b_{p-1} + b_0 - 1 \\ \beta_i = b_i + b_{i-1} \end{cases} \quad (i = 1, \dots, p-1).$$

System (10) is the system of $p + 1$ linear equation with $p + 1$ unknown elements

$$2a, \beta_0, \beta_1, \dots, \beta_{p-1}.$$

Since the determinante Δ of this system by assumption is different than zero, system (10) has only the trivial solution

$$(12) \quad a = 0, \beta_i = 0 \quad (i = 0, 1, \dots, p-1).$$

System (12) reduce to a system of linear equation which for p odd number gives (7). For p an even number she system is contradictory.

REMARK 1. We note that for the case $p \geq 2$ and p an even number $\varphi(x)$ of the form (8) is not a solution of (1). Namely, we have

$$\varphi(x) + \varphi(f(x)) = 0$$

which contradics (5) in which $\Delta \neq 0$.

This proves the theorem.

We note the following: If $F(x)$ has properties (4), then the series (2) obvioulsy diverges since

$$a_n(x) = (-1)^n \{Ff^n(x) - F(b)\}$$

doesn't converge to zero as $n \rightarrow \infty$.

Because of this, it is interest to ask if there exists a regular method of summability $T = (a_{kk})$ with which the series (2) wil be summable to T -sum $\varphi(x)$, that is the solution of the considered equation, taking into account that $F(x)$ has properties in (4). In the connection with that we have

THEOREM 2. *Let the following condition be satisfied:*

- a) *function F has properties (4) and $p > 2$ is odd number,*
- b) *the series (2) is T -sumable to sum $\varphi(x)$, where $T = (a_{k,n})$ is a regular matrix transformation*
- c) *the limit*

$$\lim_{k \rightarrow \infty} B_r^k \text{ exists for each } r \in \{0, 1, \dots, 2p-2\},$$

where

$$B_r^k = \sum_{n=1}^{\infty} a_{k,n,2p+r}, \quad r \in \{0, 1, \dots, 2p-2\}.$$

For the functions $\varphi(x)$ to be a solution of the equation (1) it is sufficient and, if F in addition satisfies the conditions in (5) it is necessary that

$$(13) \quad a = 0, \quad b_i = \frac{1}{2}(-1)^i, \quad i \in \{0, 1, \dots, p-1\}$$

and solution $\varphi(x)$ is given by (6).

PROOF. Because of theorem 1, it is sufficient to prove that, under the conditions of the present theorem, functions $\varphi(x)$ has the form (6).

Namely, under conditions of the present theorem we have

$$s_n(x) = -\frac{1}{2}(-1)^n F(b) + \sum_{v=0}^n (-1)^v F_v \quad \text{with } F_v = F(f^v(x)),$$

$$s'_k(x) = \frac{1}{2}F(b) \left\{ \sum_{n=0}^{\infty} a_{k,2n} - \sum_{n=0}^{\infty} a_{k,2k+1} \right\} + \sum_{n=0}^{\infty} a_{kn} \left(\sum_{v=0}^n (-1)^v F_v \right).$$

If we put

$$n = 2p \cdot m_n + r_n \quad (r_n = 0, 1, \dots, 2p-1; \quad m_n \in \{0, 1, 2, \dots\}),$$

it is easy to prove

$$\sum_{v=0}^n (-1)^v F_v = \begin{cases} \sum_{v=0}^{r_n} (-1)^v F_v, & p \text{ is odd number } r_n \in \{0, 1, \dots, 2p-1\}. \\ 2m_n \sum_{v=1}^{p-1} (-1)^v F_v + \sum_{v=0}^{r_n} (-1)^v F_v, & p \text{ is even number} \end{cases}$$

Now for p odd number

$$s'_k(x) = -\frac{1}{2}F(b) \left\{ \sum_{n=0}^{\infty} a_{k,2n} - \sum_{n=0}^{\infty} a_{k,2n+1} \right\} + \sum_{n=0}^{\infty} a_{k,2pm_n+r_n} \left(\sum_{v=0}^n (-1)^v F_v \right), \quad r_n \in \{0, 1, \dots, 2p-1\},$$

$$\varphi(x) = \lim_{k \rightarrow \infty} s'_k(x) = -\frac{1}{2}F(b) \{B_0 + B_2 + \dots + B_{2p-2} + B_1 - B_3 - \dots - B_{2p-1}\} + \sum_{r=0}^{2p-1} B_r \left(\sum_{v=0}^r (-1)^v F_v \right),$$

where

$$B_r^k = \sum_{n=0}^{\infty} a_{k,2pn+r}, \quad r \in \{0, 1, \dots, 2p-2\},$$

so that

$$\sum_{n=0}^{\infty} a_{k,2n} = B_0^k + B_2^k + \cdots + B_{2p-2}^k, \quad \sum_{n=0}^{\infty} a_{k,2n+1} = B_1^k + B_3^k + \cdots + B_{2p-1}^k,$$

and

$$B_r = \lim_{k \rightarrow \infty} \bar{B}_r^k, \quad r \in \{0, 1, \dots, 2p-2\}$$

and

$$\lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} a_{k,2n} = B_0 + B_2 + \cdots + B_{2p-2}, \quad \lim_{k \rightarrow \infty} \sum_{n=0}^{\infty} a_{k,2n+1} = B_1 + B_3 + \cdots + B_{2p-1}.$$

Because of this, taking into account (4), after a long but straight forward calculation we get

$$\varphi(x) = a + b_0 F_0 + b_1 F_1 + b_2 F_2 + \cdots + b_{p-1} F_{p-1},$$

where the coefficients a, b_i ($i = 0, 1, \dots, p-1$) are given by

$$a = \frac{1}{2} F(b) (B_1 + B_3 + \cdots + B_{2p-1} - B_0 - B_2 - \cdots - B_{2p-2})$$

$$b_j = (-1)^j \sum_{i=0}^{p-1} B_{i+j} \quad (j = 0, 1, 2, \dots, p-1).$$

Therefore $\varphi(x)$ is of the form (6).

REMARK 2. We note that the case $p = 1$, in a more general form, is included in theorem I of [11].

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