PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 29 (43), 1981, pp. 89–96

ON SOME CLASSES OF LINEAR EQUATIONS, IV

Jovan D. Kečkić

1. Introduction

Let V be a commutative algebra over $\mathbf{R}$ (or $\mathbf{C}$), and let L_1 and L_2 belinear operators on V satisfying the conditions

$$L_i(uv) = uL_iv \text{ iff } u \in \ker L_i \qquad (i = 1, 2). \tag{1}$$

Consider the linear equation

$$P(L_1, L_2)u = 0 \tag{2}$$

where P is a two variable polynomial over $\mathbf{R}$ (or $\mathbf{C}$) and $u \in V$.

Suppose that there exists an element $v \in V$ which is a characteristic vector of L_1 and the same time, belongs to the kernel of L_2; and similarly, that there exists an element $w \in V$ which is a characteristic vector of L_2 and belongs to the kernel of L_1. In other words, we suppose the existence of elements v, $w \in V$ such that

$$L_1v = \lambda v, \quad L_2v = 0, \tag{3}$$
$$L_1w = 0, \qquad L_2w = \mu w \tag{4}$$

where $\lambda, \mu \in \mathbf{R}$ (or $\mathbf{C}$).

Put $u = vw$. Then, in virtue of (1), (3) and (4), we easily verify that

$$L_1^k u = \lambda^k u, \quad L_2^k u = \mu^k u, \quad L_1^p L_2^q = \lambda^p \mu^q u \qquad (k, p, q \in \mathbf{N}).$$

Hence, if v, w, λ, μ are defined by (3) and (4), we see that $u = vw$ is a solution of the equation (2) provided that

$$P(\lambda, \mu) = 0. \tag{5}$$

This simple observation reduces the problem of finding a *particular* solution of the equation (2) to the problem of solving the algebraic equation (5). However, the equation (5) may have an infinity of solutions, which means that it is, in certain cases, possible to obtain an infinity of solutions of (2). Some times those solutions can be combined to yield the *general* solution of the considered equation. Namely, suppose that v_λ and w_μ $(\lambda,\mu\in\mathbf{R})$ are such that

$$L_1v_\lambda=\lambda v_\lambda,\ \ L_1w_\mu=0,\ \ L_2v_\lambda=0,\ \ L_2w_\mu=\mu w_\mu\ \ (\text{for all } \lambda,\mu\in\mathbf{R}). \tag{6}$$

If the equation (5) implies $\mu=f(\lambda)$, then

$$u=v_\lambda w_{f(\lambda)}$$

is a solution of (2) for all $\lambda\in\mathbf{R}$, and hence we can combine those solutions into a formal series

$$u=\sum_\lambda C_\lambda v_\lambda w_{f(\lambda)} \tag{7}$$

which, can, in certain cases, be summed up, the sum of (7) being the actual general solution of the equation (2).

In this paper we shall give a number of examples of first order equations of the form (2).

2. Examples of first order equations

If P is a first degree polynomial, then the equation (2) becomes

$$(aL_1+bL_2+cI)u=0 \qquad (a,b,c\in\mathbf{R}\ \text{or}\ \mathbf{C}) \tag{8}$$

where I is the identity mapping.

The corresponding algebraic equation is

$$a\lambda+b\mu+c=0, \tag{9}$$

and we, naturally, suppose that the coefficients a and b are not both zero.

Example 1: *Partial differential equations.* Let V be the algebra of all real differentiable functions in two real variables. Let $L_1\frac{\delta}{\delta x}$, $L_2=\frac{\delta}{\delta y}$. Then $v_\lambda=e^{\lambda x}$, $w_\mu=e^{\mu y}$ satisfy the equations (6) for all $\lambda,\mu\in\mathbf{R}$. The equation (8)in this case becomes

$$au_x+bu_y+cu=0. \tag{10}$$

Supposing that $b\neq 0$, we obtain the following solution of (10):

$$u=e^{\lambda x}e^{-(a\lambda+c)y/b}$$

where $\lambda \in \mathbf{R}$ is arbitrary. Combining the obtained solutions into a formal series of the form (7), we find

$$u = \sum_{\lambda} C_\lambda e^{\lambda x} e^{-(a\lambda + c)y/b},$$

i.e.

$$u = e^{-\frac{c}{b}y} \sum_{\lambda} C_\lambda e^{\frac{\lambda}{b}(bx - ay)}, \tag{11}$$

where C_λ are arbitrary constans. This suggests that the sum which appears in (11) is an arbitrary (differentiable) function of $bx - ay$, giving *general* solution of (10):

$$u(x, y) = e^{-\frac{c}{b}y} F(bx - ay),$$

where F is an arbitrary differetiable function.

Example 2: *Partial difference equations.* Let V be the algebra of all real functions in two real variables x and y, and let $L_1 = \Delta_x$, $L_2 = \Delta_y$, where

$$\Delta_x u(x, y) = u(x + 1, y) - u(x, y), \quad \Delta_y u(x, y) = u(x, y + 1) - u(x, y)$$

Then $v_\lambda = (1 + \lambda)^x$, $w_\mu = (1 + \mu)^y$ satisfy the equations (6) for all $\lambda, \mu \in \mathbf{R}$. The equation (8) in this case becomes

$$a\Delta_x u + b\Delta_y u + cu = 0,$$

i.e.

$$au(x + 1, y) + bu(x, y + 1) + (c - a - b)u(x, y) = 0. \tag{12}$$

Supposing that $b \neq 0$, we obtain the following solution of (12):

$$u = (1 + \lambda)^x \left(1 - \frac{1}{b}(a\lambda + c)\right)^y$$

with $\lambda \in \mathbf{R}$ arbitrary. Hence

$$u(x, y) = \sum_{\lambda} C_\lambda (1 + \lambda)^x \left(1 - \frac{a}{b}\lambda - \frac{c}{b}\right)^y \tag{13}$$

is also a formal solution of (12) for arbitrary constans C_λ. We distinguish between three cases:

(i) $a+b \neq c; \quad a=0; \quad b \neq 0$. Then (13) becomes

$$\begin{aligned} u(x,y) &= \sum_{\lambda} C_\lambda (1+\lambda)^x \left(1-\frac{c}{b}\right)^y = \left(\frac{b-c}{b}\right)^y \sum_{\lambda} C_\lambda (1+\lambda)^x \\ &= \left(\frac{b-c}{b}\right)^y f(x), \quad \text{where } f \text{ is arbitrary.} \end{aligned}$$

(ii) $a+b=c; \quad ab \neq 0$. Then (13) becomes

$$\begin{aligned} u(x,y) &= \sum_{\lambda} C_\lambda (1+\lambda)^x \left(-\frac{a}{b}-\frac{a}{b}\lambda\right)^y = \left(-\frac{a}{b}\right)^y \sum_{\lambda} C_\lambda (1+\lambda)^{x+y} \\ &= \left(-\frac{a}{b}\right)^y f(x+y), \quad \text{where } f \text{ is arbitrary.} \end{aligned}$$

(iii) $a+b \neq c; \quad b \neq 0$. Then (13) becomes

$$\begin{aligned} u(x,y) &- \left(-\frac{a}{b}\right)^y \sum_{\lambda} C_\lambda (1+\lambda)^x \left(\frac{c-a-b}{a}+1+\lambda\right)^y \\ &= \left(-\frac{a}{b}\right)^y \sum_{\lambda} C_\lambda (1+\lambda)^x \sum_{v=0}^{y} \binom{y}{v} \left(\frac{c-a-b}{a}\right)^{y-v} (1+\lambda)^v \\ &= \left(\frac{a+b-c}{b}\right)^y \sum_{v=0}^{y} \binom{y}{v} \left(\frac{a}{c-a-b}\right)^v f(x+v), \end{aligned}$$

where f is arbitrary.

The obtained solutions are general.

Example 3: *Differential-difference equations.* Let $V = \{u_n(x) \mid n \in \mathbf{N},\ x \in \mathbf{R}\}$ and define L_1 and L_2 by:

$$L_1 u_n(x) = u_{n+1}(x) - u_n(x), \quad L_2 u_n(x) = u_n'(x).$$

This leads to the equation

$$a u_{n+1}(x) + b {u_n}'(x) + (c-a) u_n(x) = 0. \tag{14}$$

Since $v_\lambda = (1+\lambda)^n$, $w_\mu = e^{\mu x}$ satisfy (6) for all $\lambda, \mu \in \mathbf{R}$, for $b \neq 0$ we obtain the following solution of (14):

$$u_n(x) = (1+\lambda)^n e^{-\frac{1}{b}(a\lambda + c)x},$$

for all $\lambda \in \mathbf{R}$. The corresponding formal series is given by

$$\begin{aligned} u_n(x) &= e^{-\frac{c}{b}x} \sum_{\lambda} C_\lambda (1+\lambda)^n e^{-\frac{a}{b}\lambda x} = e^{-\frac{c}{b}x} \sum_{v} C_v \left(1-\frac{b}{a}v\right)^n e^{vx} \\ &= e^{-\frac{c}{b}x} \sum_{v} C_v \left[1 - \binom{n}{1}\frac{b}{a}v + \binom{n}{2}\frac{b^2}{a^2}v^2 - \cdots + (-1)^n \frac{b^n}{a^n} v^n\right] e^{vx}. \end{aligned}$$

Hence, putting

$$\sum_v C_v e^{vx} = f(x),$$

where f is an arbitrary n times differentiable function, we arrive at the following solution of (14):

$$u_n(x) = e^{-\frac{c}{b}x}\left[f(x) - \binom{n}{1}\frac{b}{a}f'(x) + \binom{n}{2}\frac{b^2}{a^2}f''(x) - \cdots + (-1)^n\frac{b^n}{a^n}f^{(n)}(x)\right].$$

EXAMPLE 4: *Functional-differential equations.* Let V be the algebra of all real differentiable functions in two real variables x and y. Define the operators L_1 and L_2 by

$$L_1u(x,y) = u(\theta x, y) - u(x,y), \;\; L_2u(x,y) = u_y(x,y)$$

where $\theta\colon \mathbf{R} \to \mathbf{R}$ is such that $\theta^2 = I$, i.e. $\theta(\theta(x)) = x$. The linear equation (8) becomes

$$au(\theta x, y) + bu_y(x,y) + (c-a)u(x,y) = 0. \tag{15}$$

As before, any function of the form $u(x,y) = e^{\mu y}$ is a characteristic vector for L_2 and also belongs to ker L_1. In order to find the characteristic vectors of L_1 we have to solve the equation

$$u(\theta x, y) = (1+\lambda)u(x,y),$$

which is possible only if $(1+\lambda)^2 = 1$, i. e. $\lambda(\lambda+2) = 0$.

For $\lambda = 0$ we find $\mu = -\frac{c}{b}$, while for $\lambda = -2$ we get $\mu = \frac{2a-c}{b}$. The characteristic vectors which correspond to $\lambda = 0$ and $\lambda = -2$ respectively, are

$$u(x,y) = f(x) + f(\theta x) \text{ and } u(x,y) = g(x) - g(\theta x),$$

where f and g are arbitraray. Hence, we obtain the following solution of the equation (15):

$$u(x,y) = [f(x) + f(\theta x)]e^{-\frac{c}{b}y} + [g(x) - g(\theta x)]e^{\frac{2a-c}{b}y}$$

where f and g are arbitrary functions.

EXAMPLE 5: *Functional-difference equations.* Let V be as in Example 2, and θ as in Example 4. Define the operators L_1 and L_2 by

$$L_1u(x,y) = u(\theta x, y) - u(x,y), \quad L_2u(x,y) = u(x, y+1) - u(x,y).$$

This leads to the equation

$$au(\theta x, y) + bu(x, y+1) + (c-a-b)u(x,y) = 0. \tag{16}$$

Again we only have two characteristic values for L_1, namely $\lambda = 0$ and $\lambda = -2$. Hence, $\mu = -\frac{c}{b}$ and $\mu = \frac{2a-c}{b}$ respectively, and we arrive at the following solution of (16):

$$u(x,y) = [f(x) + f(\theta x)] \left(1 - \frac{c}{b}\right)^y + [g(x) - g(\theta x)] \left(1 - \frac{2a-c}{b}\right)^y,$$

where f and g are arbitrary functions.

EXAMPLE 6: *Functional equations.* Let V be as in Example 2, and let θ, $\varphi\colon \mathbf{R} \to \mathbf{R}$ be such that $\theta^2 = I$, $\varphi^3 = I$ $(Ix = x)$. Define the operators L_1 and L_2 by

$$L_1 u(x,y) = u(\theta x, y) - u(x,y), \quad L_2 u(x,y) = u(x, \varphi y) - u(x,y).$$

The linear equation (8) becomes

$$au(\theta x, y) + bu(x, \varphi, y) + (c - a - b)u(x,y) = 0. \tag{17}$$

There are only two possible characteristic values λ for L_1: 0 and -2, and only one (real) characteristic value μ for L_2: 0. Since λ and μ are tied by the equation (9), this equation has to be satisfied by the following two pairs: $(0,0)$ and $(-2,0)$, which yields $c = 0$ and $c = 2a$, respectively.

Hence, we can obtain solutions of the following equations

$$au(\theta x, y) + bu(x, \varphi y) - (a+b)u(x,y) = 0, \tag{18}$$

$$au(\theta x, y) + bu(x, \varphi y) + (a-b)u(x,y) = 0. \tag{19}$$

They are

$$u(x,y) = (f(x) + f(\theta x))(g(y) + g(\varphi y) + g(\varphi^2 y)), \tag{20}$$
$$u(x,y) = (f(x) - f(\theta x))(g(y) + g(\varphi y) + g(\varphi^2 y)), \tag{21}$$

respectively, where f and g are arbitrary functions.

REMARK 1. If we put $f(x)g(y) = F(x,y)$, then (20) and (21) can be written as

$$\begin{aligned} u(x,y) = F(x,y) + F(x,\varphi y) &+ F(x,\varphi^2 y) \\ &+ F(\theta x, y) + F(\theta x, \varphi y) + F(\theta x, \varphi^2 y) \end{aligned} \tag{22}$$

$$\begin{aligned} u(x,y) = F(x,y) + F(x,\varphi y) &+ F(x,\varphi^2 y) \\ &- F(\theta x, y) - F(\theta x, \varphi y) + F(\theta x, \varphi^2 y), \end{aligned} \tag{23}$$

respectively.

It is easily verified that (22) and (23) satisfy the equations (18) and (19), when F is an arbitrary function. Moreover, is can be shown that the obtained solutions are *general.* Indeed, the equation (18) is equivalent to the equation

$$u(x,y) = \frac{1}{6}\left(u(x,y) + u(x,\varphi y) + u(x,\varphi^2 y) + u(\theta x, y) + u(\theta x, \varphi y) + (\theta x, \varphi^2 y)\right),$$

which is a reproductive equation, i.e. has the form

$$u = Au, \quad \text{with} \quad A^2 = A.$$

Hence, see for example [1], its general solution is

$$u(x,y) = A\Pi(x,y),$$

where Π is arbitrary, which is precisely (22) with $6F = \Pi$.

Similar constutions hold for the solution (23) of the equation (19).

REMARK 2. It can also be shown that the equations (18) and (19) are the only equations of the form (17) which have nontrivial solutions.

3. A remark on first order equations

The exposed method gives only one or more *particular* solutions of the considered equation. However, in Examples 1, 2 and 6, using those particular solutions we arrived at the *general* solutions. Moreover, the solution of the equation (14) from Example 3 is also called the general solution in book [2], though no proof is given. Some other investigations, which are not the subject of this paper, lead us to conjecture that the solutions given in Examples 4 and 5 are also general.

4. A note on some second order equations

The method given here can clearly be applied to higher order equations, but the corresponding algebraic equation will be more complicated. In papere [3] and [4] we applied this method to second order partial differential and difference equations.

We shall briefly consider some examples of the second order equation

$$(L_1L_2 + aL_1 + bL_2 + cI)u = 0 \qquad (a,b,c \in \mathbf{R}) \tag{24}$$

with the corresponding algebraic equation

$$\lambda\mu + a\lambda + b\mu + c = 0.$$

If V, L_1, L_2 are defined as in Examples 3, 4, 5 and 6, respectively, we obtain the following equations

$$u_{n+1}{}'(x) + au_{n+1}(x) + (b-1)u_n{}'(x) + (c-a)u_n(x) = 0, \tag{25}$$

$$(26)\qquad u_y(\theta x, y) + au(\theta x, y) + (b-1)u_y(x, y) + (c-a)u(x, y) = 0,$$
$$(27)\quad u(\theta x, y+1) + (a-1)u(\theta x, y) + (b-1)u(x, y+1) + (c-a-b+1)u(x, y) = 0,$$
$$(28)\quad u(\theta x, \varphi y) + (a-1)u(\theta x, y) + (b-1)u(x, \varphi y) + (c-a-b+1)u(x, y) = 0.$$

The equation (25) is formally satisfied by

$$(29)\qquad u_n(x) = \sum_{\lambda} C_\lambda (1+\lambda)^n e^{\frac{a\lambda+c}{\lambda+b}x}$$

and this series can, in some cases, be summed. For example, if $c = ab$ then (29) reduces to $u_n(x) = f(n)e^{-ax}$, where f is arbitrary.

For the equation (26) we obtain the solutions:

$$\begin{aligned}
&u(x,y) = (f(x) + f(\theta x)e^{-\frac{c}{b}y} + (g(x) - g(\theta x))e^{\frac{2a-c}{b-a}y} && \text{if}\quad b(b-2) \neq 0,\\
&u(x,y) = F(x,y) + F(\theta x, y) + (g(x) - g(\theta x))e^{-ay} && \text{if}\quad b = c = 0,\\
&u(x,y) = (g(x) - g(\theta x))e^{\frac{c-2a}{2}y} && \text{if}\quad b = 0,\ \ c \neq 0,\\
&u(x,y) = (f(x) + f(\theta x))e^{-ay} + G(x,y) - G(\theta x, y) && \text{if}\quad b = 2,\ \ c = 2a,\\
&u(x,y) = (f(x) + f(\theta x))e^{-\frac{c}{2}y} && \text{if}\quad b = 2,\ \ c \neq 2a,
\end{aligned}$$

where f, g, F, G are arbitrary functions.

Similar results hold for the equation (27).

Finally, regarding the equation (28) we conclude that we can obtain solutions only for the following two equations

$$\begin{aligned}
&u(\theta x, \varphi y) + (a-1)u(\theta x, y) + (b-1)u(x, \varphi y) + (1-a-b)u(x, y) = 0,\\
&u(\theta x, \varphi y) + (a-1)u(\theta x, y) + (b-1)u(x, \varphi y) + (1+a-b)u(x, y) = 0.
\end{aligned}$$

The solutions are again (22) and (23), respectively, where F is an arbitrary function.

REFERENCES

[1] S. B. Prešić, *Une Classe d'équations matricielles et l'équation fonticionelle* $f^2 = f$. Publ. Inst. Math. (Beograd) **8** (22) (1968), 143–148.

[2] M. R. Spiegel, *Finite differences and difference equations*. New York 1971.

[3] J. D. Kečkić, *On integration of linear partial differential equations with constant coefficients*. Math. Balkanica **1** (1971), 134–139.

[4] J. D. Kečkić, *Analogies between differential and difference equations II: First and non-parabolic second order partial equations with constant coefficients*. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **634**–No. **677** (1979), 192–196.

Tikveška 2
11000 Beograd