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LIMIT THEOREMS FOR TWO-UNIT STANDBY REDUNDANT SYSTEMS WITH RAPID REPAIR AND RAPID PREVENTIVE MAINTENANCE

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Formulation of the problem. During the last few years several papers appeared, dealing the behaviour of the redundant systems with repair and preventive maintenance (preventive repair).

In reality, the repair time and preventive repair time in a system is by far shorter then the period without failure of a unit, and so, it is natural to investigate how the redundant systems behave when repair and preventive repair are rapid. In this paper we discuss the two-unit standby system and we impose the following conditions on the work of the system:

1. the standby unit is unloaded;

2. after repair and preventive repair completion a unit recovers its function perfectly;

3. after repair or preventive repair a unit is in the standby state;

4. we assume that the switchover times, from the failure to the repair, from the repair completion to the standby state, and from the standby state to the operative state, of each unit are all instantaneous, and such are the switchover times occuring in the inspection too;

5. the standby unit begins to work immediately when the working unit goes from the operative state to repair or preventive repair;

6. the repair time distribution and the preventive repair time distribution are independant of the failure time distribution or the inspection time distribution;

7. the failure time distribution, the repair time distribution, the inspection time distribution and the preventive repair time distribution are, respectively, $F(x)$, $G(x)$, $U(x)$ and $V(x)$.

We shall deal with three types of the inspection strategies: rigid, sliding and economical inspection strategy. When the inspection strategy is rigid, a unit undergoes inspection within certain (in general random) interval of time, independantly of the state of the other unit. When the inspection strategy is sliding, operating unit submits preventive repair only when the space unit is in the standby state. If the spare unit is not in the standby, the unit which needs preventive repair goes on working until the spare unit is repared, or until the failure of the two-unit system. When the inspection strategy is economical, and the moment for the inspection of the working unit comes when the spare unit is not in order, then the working unit continues with work until its failure.

The effect of the inspection strategies on two-unit systems with the three mentioned types of stategies was investigated in [1] and [2]. Besides, Laplace transforms of a time without failure distributions for the corresponding strategies were given there. In our paper we shall use those formulas, but we shall write them in a slightly different way – actually, instead of Laplace transforms we shall use Laplace-Stieltjes transforms.

Let $\Phi_*(x)$ be a time without failure distribution function of our system and $\mathcal{S}_*(s)$ the corresponding Laplace-Stieltjes transform, where instead of the symbol $*$ we shall have one of the letters r, sl, e, which correspond to the case of rigid, sliding and economical inspection strategy. Then [1], [2]

$$
\begin{aligned}
(1)\qquad \mathcal{S}_r(s) =& d_1(s) + d_2(s) - (1 - d_1(s) - d_2(s))(d_1(s)(1 - c_2(s) + c_1(s)) + \\
& d_2(s)(1 - b_2(s) + c_1(s)))((1 - b_1(s))(1 - c_2(s)) - b_2(s)c_1(s))^{-1} \\
\mathcal{S}_{sl}(s) =& ((d_1(s) + d_2(s))((1 - b_1(s))(1 - c_2(s) - e_2(s)) - b_2(s)(c_1(s) + \\
& e_1(s)) + d_1(s)(1 + c_1(s) - c_2(s) + e_1(s) - e_2(s))(-1 + g_1(s) \\
& + c_1(s) + e_1(s)) + d_2(s)(1 - b_1(s) + b_2(s))(-1 + g_2(s) + \\
& c_2(s) + e_2(s)))((1 - b_1(s))(1 - c_2(s) - e_2(s)) - b_2(s))(c_1(s) \\
& + e_1(s)))^{-1} \\
\mathcal{S}_e(s) =& ((d_1(s) + d_2(s))((1 - \alpha_1(s) - b_1(s))(1 - c_2(s)) - (\alpha_2(s) + \\
& b_2(s))c_1(s)) - (d_1(s)(1 - c_2(s)) + d_2(s)(\alpha_2(s) + b_2(s))(1 - \\
& d_1(s) - \alpha_1(s) - c_1(s) - h_1(s)) - (d_1(s)c_1(s) + d_2(s)(1 - \\
& \alpha_1(s) - b_1(s))(1 - d_1(s) - c_2(s) - \alpha_2(s) - h_2(s)))(1 - \\
& \alpha_1(s) - b_1(s))(1 - c_2(s)) - (\alpha_2(s) + b_2(s))c_1(s))^{-1}
\end{aligned}
$$

where

$$\mathcal{S}_*(s) = \int_0^\infty e^{-sx} d\Phi_*(x);\ d_1(s) = \int_0^\infty e^{-sx}\overline{U(x)}dF(x)^1;\ d_2(s) = \int_0^\infty e^{-sx}\overline{F(x)}dU(x)$$

$$b_1(s) = \int_0^\infty e^{-sx}G(x)\overline{U(x)}dF(x);\quad b_2(s) = \int_0^\infty e^{-sx}V(x)\overline{U(x)}dF(x);$$

$$c_1(s) = \int_0^\infty e^{-sx}G(x)\overline{F(x)}dU(x);\quad c_2(s) = \int_0^\infty e^{-sx}V(x)\overline{F(x)}dU(x);$$

[1] $\overline{U(x)} = 1 - U(x)$

$$e_1(s) = \int_0^\infty e^{-sx}\overline{F(x)}U(x)dG(x);\quad e_2(s) = \int_0^\infty e^{-sx}\overline{F(x)}U(x)dV(x);$$

$$g_1(s) = \int_0^\infty e^{-sx}\overline{G(x)U(x)}dF(x);\quad g_2(s) = \int_0^\infty e^{-sx}\overline{V(x)U(x)}dF(x);$$

$$\alpha_1(s) = \int_0^\infty e^{-sx}\int_0^\infty U(y)dG(y)dF(x);\quad \alpha_2(s) = \int_0^\infty e^{-sx}\int_0^\infty U(y)dV(y)dF(y);$$

$$h_1(s) = \int_0^\infty e^{-sx}\overline{G(x)}U(x)dF(x);\quad h_2(s) = \int_0^\infty e^{-sx}\overline{V(x)}U(x)dF(x).$$

The limiting time without failure distribution function of the two-unit system with rapid repair without inspection was given in [3]. We are going to investigate the limiting distribution of time without failure of two-unit systems with already mentioned three types of inspection strategies, under the assumption that repair and preventive repair are rapid. So, let us suppose that the distrubution functions of the life of a unit $F(x)$ and of the time until the beginning of inspection $U(x)$ are fixed, and the repair time distribution function $G_n(x)$ and the preventive repair time distribution function $V_n(x)$ change with the sequence $\{n\}$ so that the following conditions are satisfied:

$$(2)\qquad \begin{aligned} &\int_0^\infty \overline{G_n(x)}dF(x) \underset{n\to\infty}{\to} 0;\quad \int_0^\infty \overline{V_n(x)}dF(x) \underset{n\to\sim}{\to} 0;\\ &\int_0^\infty \overline{G_n(x)}dU(x) \underset{n\to\infty}{\to} 0;\quad \int_0^\infty \overline{V_n(x)}dU(x) \underset{n\to\sim}{\to} 0. \end{aligned}$$

The conditions (2) we call the conditions of rapid repair and rapid preventive repair. According to those conditions, we shall give indexes to the Laplace-Stieltjes transforms where $V_n(x)$ or $G_n(x)$ appear.

Results. Let us denote by τ the random variable which corresponds to the time withour failure of the two-unit system. Then the following theorems hold:

Theorem 1. *In the two-unit system with rigid inspection strategy under the conditions* (2) *of rapid repair and rapid preventive repair*

$$\lim_{n\to\infty} P\{\alpha_n\tau < t\} = 1 - \exp(-t/M)$$

where $M = \int_0^\infty x\overline{U(x)}dF(x) + \int_0^\infty x\overline{F(x)}dU(x)$, *and* α_n *is a sequence tending to zero,*

$$\alpha_n = (1 - b_{1n}(0))(1 - c_{2n}(0)) - b_{2n}(0)c_{1n}(0).$$

Let us notice that the mathematical expectation of the limiting distribution is $M = \int_0^\infty x dF(x) + \int_0^\infty x dU(x) - \int_0^\infty x d(F(x)U(x))$. The last summand is the mathematical expectation of a maximum of the following two random variables: the time until the failure of a unit and the time until the moment for inspection, and so $M \le \max\left\{\int_0^\infty x dF(x),\ \int_0^\infty x dU(x)\right\}$.

THEOREM 2. *In the two-unit system with sliding inspection strategy under the conditions* (2) *of rapid and rapid preventive repair*

$$\lim_{n\to\infty} P\{\alpha_n \tau < t\} = 1 - \exp(-t/M)$$

where $M = \int_0^\infty x\overline{U(x)}dF(x) + \int_0^\infty x\overline{F(x)}dU(x)$, *and* α_n *is a sequence tending to zero,*

$$\alpha_n = (1 - b_{1n}(0))(1 - c_{2n}(0) - e_{2n}(0)) - b_{2n}(0)(c_{1n}(0) + e_{1n}(0)).$$

THEOREM 3. *In the two-unit system with economical inspection strategy under the conditions* (2) *of rapid repair and rapid preventive repair*

$$\lim_{n\to\infty} \{\alpha_n \tau < t\} = 1 - \exp(t/M)$$

where $M = \int_0^\infty x\overline{U(x)}dF(x) + \int_0^\infty x\overline{F(x)}dU(x)$, *and* α_n *is a sequence tending to zero.*

$$\alpha_n = (1 - \alpha_{1n}(0) - b_{1n}(0))(1 - c_{2n}(0)) - (\alpha_{2n}(0) + b_{2n}(0))c_{1n}(0).$$

REMARK. The following relations exist between the coefficients which correspond to different inspection strategies: $\alpha_n^r \ge \alpha_n^e$, $\forall_n$ and $\alpha_n^{sl} \le \alpha_n^r$, $\forall_n$ (where α_n^r, α_n^{sl}, α_n^e are, respectively, coefficients corresponding to rigid, sliding and economical inspection strategy). On the other hand, between α_n^{sl} and α_n^e no such relation exists – which is obvious from the fact that $\alpha_n^e - \alpha_n^{sl} = \int_0^\infty \overline{G_n(x)}dF(x) \int_0^\infty \overline{F(x)}U(x)dV_n(x) - \int_0^\infty \overline{V_n(x)}dF(x) \int_0^\infty \overline{F(x)}U(x)dG_n(x)$. Roughly speaking, those relations show that the time without failure for the system with rigid inspection strategy is shorter then the time time without failure for systems with sliding or economical strategy.

It follows from the Theorems 1., 2. and 3. that in the case of rapid repair and rapid preventive repair we obtain the same limiting distribution for the two-unit systems with rigid, sliding and economical inspection strategy, which means that the limiting case the type of inspection strategy has no influence on the length of time without failure in our system.

The proofs of the Theorems 1., 2. and 3. are analogous (and cumbersome) and therefore we shall give here only the proof of the Theorem 1.

Proof of the Theorem 1. With some simple transformations, we can write the function $\mathcal{S}_{rn}(s)$ in the following way

$$
\begin{aligned}
&\mathcal{S}_{rn}(s) = \\
&\Bigg\{\left(\int_0^\infty e^{-sx}\overline{U(x)}dF(x) + \int_0^\infty e^{-sx}\overline{F(x)}dU(x)\right)\left(\int_0^\infty e^{-sx}\overline{G_n(x)U(x)}dF(x) + \right. \\
&\int_0^\infty e^{-sx}\overline{V_n(x)F(x)}dU(x) - \int_0^\infty e^{-sx}V_n(x)\overline{F(x)}dU(x)\int_0^\infty e^{-sx}\overline{G_n(x)U(x)}dF(x) + \\
&\int_0^\infty e^{-sx}V_n(x)\overline{U(x)}dF(x)\int_0^\infty e^{-sx}G_n(x)\overline{F(x)}dU(x) + \int_0^\infty e^{-sx}\overline{F(x)}dU(x)\int_0^\infty e^{-sx}\overline{(V_n(x) -} \\
&\overline{G_n(x))U(x)}dF(x)) + \int_0^\infty e^{-sx}\overline{U(x)}dF(x)\int_0^\infty e^{-sx}\overline{(G_n(x)} - \overline{V_n(x))F(x)}dU(x) + \\
&\int_0^\infty e^{-sx}\overline{F(x)}dU(x)\int_0^\infty e^{-sx}\overline{(G_n(x)} - \overline{V_n(x))U(x)}dF(x)\Bigg\}\Bigg\{1 - \int_0^\infty e^{-sx}G_n\overline{(x)U(x)}dF(x) \\
&- \int_0^\infty e^{-sx}V_n(x)\overline{F(x)}dU(x) + \int_0^\infty e^{-sx}G_n(x)\overline{U(x)}dF(x)\int_0^\infty e^{-sx}V_n\overline{F(x)}dU(x) - \\
&\int_0^\infty e^{-sx}V_n(x)\overline{U(x)}dF(x)\int_0^\infty e^{-sx}G_n(x)\overline{F(x)}dU(x)\Bigg\}^{-1}
\end{aligned}
$$

Let us denote by $p_{1n}(s)$ and $p_n(s)$ the denominator and the numerator of $\mathcal{S}_{rn}(s)$ respectively. Then $p_n(0) = p_{1n}(0)$, because of $\mathcal{S}_{rn}(0) = 1$. That is also obvious from (1), owing to the fact that $\int_0^\infty \overline{U(x)}dF(x) + \int_0^\infty \overline{F(x)}dU(x) = 1$ (or $d_1(0) + d_2(0) = 1$). Let us put $\alpha_n = p_n(0) = p_{1n}(0)$.

Lemma 1.

$$\lim_{n\to\infty}\frac{p_n(\alpha_n s)}{\alpha_n} = 1$$

uniformly on s on every limited interval.

Proof. Let us write the numerator of $\mathcal{S}_{rn}(\alpha_n s)$ in the following way:

$$
\begin{aligned}
&p_n(\alpha_n s) = \\
&\quad\int_0^\infty e^{-\alpha_n sx}\overline{G_n(x)U(x)}dF(x)\left(\left(\int_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x) + \int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\right)\right.
\end{aligned}
$$

$$\Bigg(1-\int_0^\infty e^{-\alpha_n sx}V_n(x)\overline{F(x)}dU(x)-\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\Bigg)+$$

$$\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\Bigg)+\int_0^\infty e^{-\alpha_n sx}\overline{G_n(x)F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}dF(x)+$$

$$\int_0^\infty e^{-\alpha_n sx}\overline{V_n(x)F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)+\int_0^\infty e^{-\alpha_n sx}\overline{V_n(x)U(x)}dF(x)$$

$$\Bigg(\Bigg(\Bigg(\int_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)+\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\Bigg)\Bigg(\int_0^\infty e^{-\alpha_n sx}G_n(x)\overline{F(x)}dU(x)+$$

$$\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\Bigg)-\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\Bigg)=$$

$$=k_1(\alpha_n s)\int_0^\infty e^{-\alpha_n sx}\overline{G_n(x)U(x)}dF(x)+k_2(\alpha_n s)\int_0^\infty e^{-\alpha_n sx}\overline{G_n(x)F(x)}dU(x)+$$

$$k_3(\alpha_n s)\int_0^\infty e^{-\alpha_n sx}\overline{V_n(x)F(x)}dU(x)+k_4(\alpha_n s)\int_0^\infty e^{-\alpha_n sx}\overline{V_n(x)U(x)}dF(x).$$

Let

$$\begin{aligned}p_n{}'(\alpha_n s)=&k_1(\alpha_n s)\int_0^\infty \overline{G_n(x)U(x)}dF(x)+k_2(\alpha_n s)\int_0^\infty \overline{G_n(x)F(x)}dU(x)+k_3(\alpha_n s)\\&\int_0^\infty \overline{V_n(x)F(x)}dU(x)+k_4(\alpha_n s)\int_0^\infty \overline{V_n(x)U(x)}dF(x)\end{aligned}$$

$$\begin{aligned}\alpha_n=&k_1(0)\int_0^\infty \overline{G_n(x)U(x)}dF(x)+k_2(0)\int_0^\infty \overline{G_n(x)F(x)}dU(x)+k_3(0)\int_0^\infty \overline{V_n(x)}\\&\overline{F(x)}dU(x)+k_4(0)\int_0^\infty \overline{V_n(x)U(x)}dF(x).\end{aligned}$$

We shall show that $\lim_{n\to\infty}\frac{\alpha_n-p_n{}'(\alpha_n s)+p_n{}'(\alpha_n{}'s)-p_n(\alpha_n s)}{\alpha_n}=0$ uniformly on s on every limited interval. First, we show that

a) $\lim_{n\to\infty}\frac{\alpha_n-p_n{}'(\alpha_n s)}{\alpha_n}=0$ uniformly on s every limited interval.

$$\alpha_n - {p_n}'(\alpha_n s) =$$

$$\int_0^\infty \overline{G_n(x)U(x)}dF(x)\left(\left(\int_0^\infty \overline{U(x)}dF(x)+\int_0^\infty \overline{F(x)}dU(x)\right)(1-\right.$$

$$\left.\left.\int_0^\infty V_n(x)\overline{F(x)}dU(x)-\int_0^\infty \overline{F(x)}dU(x)\right)+\int_0^\infty \overline{F(x)}dU(x)\right)+\int_0^\infty \overline{G_n(x)}$$

$$\overline{F(x)}dU(x)\int_0^\infty \overline{U(x)}dF(x)+\int_0^\infty \overline{V_n(x)F(x)}dU(x)\int_0^\infty \overline{F(x)}dU(x)+$$

$$\int_0^\infty \overline{V_n(x)U(x)}dF(x)\left(\left(\left(\int_0^\infty \overline{U(x)}dF(x)+\int_0^\infty \overline{F(x)}dU(x)\right)\left(\int_0^\infty \overline{G_n(x)}\right.\right.\right.$$

$$\left.\left.\left.\overline{F(x)}dU(x)+\int_0^\infty \overline{F(x)}dU(x)\right)-\int_0^\infty \overline{F(x)}dU(x)\right)-\int_0^\infty \overline{G_n(x)U(x)}dF(x)$$

$$\left(\left(\left(\int_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)+\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\right)\left(1-\int_0^\infty e^{-\alpha_n sx}V_n(x)\right.\right.\right.$$

$$\left.\left.\overline{F(x)}dU(x)-\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)+\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\right)-\int_0^\infty \overline{G_n(x)}\right.$$

$$\overline{F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)-\int_0^\infty \overline{V_n(x)F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}$$

$$\overline{F(x)}dU(x)-\int_0^\infty \overline{V_n(x)U(x)}dF(x)\left(\left(\left(\int_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)+\right.\right.\right.$$

$$\left.\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\right)\left(\int_0^\infty e^{-\alpha_n sx}G_n(x)\overline{F(x)}dU(x)+\right.$$

$$\left.\left.\left.\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\right)-\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\right)=$$

$$=\int_0^\infty \overline{G_n(x)U(x)}dF(x)\left(\int_0^\infty (1-e^{-\alpha_n sx})\overline{U(x)}dF(x)+2\int_0^\infty (1-e^{-\alpha_n sx})\right.$$

$$\overline{F(x)}dU(x)-\int\limits_0^\infty(1-e^{-\alpha_n sx})\overline{U(x)}dF(x)\int\limits_0^\infty V_n(x)\overline{F(x)}dU(x)-$$

$$\int\limits_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)\int\limits_0^\infty(1-e^{-\alpha_n sx})V_n(x)\overline{F(x)}dU(x)-\int\limits_0^\infty(1-$$

$$e^{-\alpha_n sx}\overline{F(x)}dU(x)\int\limits_0^\infty V_n(x)\overline{F(x)}dU(x)-\int\limits_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\int\limits_0^\infty(1-$$

$$e^{-\alpha_n sx})V_n(x)\overline{F(x)}dU(x)-\int\limits_0^\infty(1-e^{-\alpha_n sx})\overline{U(x)}dF(x)\int\limits_0^\infty\overline{F(x)}dU(x)-$$

$$\int\limits_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)\int\limits_0^\infty(1-e^{-\alpha_n sx})\overline{F(x)}dU(x)-\int\limits_0^\infty(1-e^{-\alpha_n sx})$$

$$\overline{F(x)}dU(x)\int\limits_0^\infty\overline{F(x)}dU(x)-\int\limits_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\int\limits_0^\infty(1-e^{-\alpha_n sx})$$

$$\overline{F(x)}dU(x)\Big)\int\limits_0^\infty\overline{G_n(x)F(x)}dU(x)\int\limits_0^\infty(1-e^{-\alpha_n sx})\overline{U(x)}dF(x)+\int\limits_0^\infty\overline{V_n(x)}$$

$$\overline{F(x)}dU(x)\int\limits_0^\infty(1-e^{-\alpha_n sx})\overline{F(x)}dU(x)+\int\limits_0^\infty\overline{V_n(x)U(x)}dF(x)\Bigg(\int\limits_0^\infty(1-$$

$$e^{-\alpha_n sx})\overline{U(x)}dF(x)\int\limits_0^\infty G_n(x)\overline{F(x)}dU(x)+\int\limits_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)\int\limits_0^\infty(1-$$

$$e^{-\alpha_n sx})G_n(x)\overline{F(x)}dU(x)+\int\limits_0^\infty(1-e^{-\alpha_n sx})\overline{F(x)}dU(x)\int\limits_0^\infty G_n(x)$$

$$\overline{F(x)}dU(x)+\int\limits_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\int\limits_0^\infty(1-e^{-\alpha_n sx})G_n(x)\overline{F(x)}dU(x)+$$

$$\int\limits_0^\infty(1-e^{-\alpha_n sx})\overline{U(x)}dF(x)\int\limits_0^\infty\overline{F(x)}dU(x)+\int\limits_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)\int\limits_0^\infty(1-$$

$$e^{-\alpha_n sx}\overline{F(x)}dU(x)+\int\limits_0^\infty(1-e^{-\alpha_n sx})\overline{F(x)}dU(x)\int\limits_0^\infty\overline{F(x)}dU+\int\limits_0^\infty e^{-\alpha_n sx}$$

$$\overline{F(x)}dU(x)\int_0^\infty(1-e^{-\alpha_n sx})\overline{F(x)}dU(x)-\int_0^\infty(1-e^{-\alpha_n sx})\overline{F(x)}dU(x)\Big)^2$$

$$\leq \alpha_n s\left(\int_0^\infty\overline{G_n(x)U(x)}dF(x)\left(\int_0^\infty x\overline{U(x)}dF(x)+2\int_0^\infty x\overline{F(x)}dU(x)+\right.\right.$$

$$\int_0^\infty x\overline{U(x)}dF(x)\int_0^\infty V_n(x)\overline{F(x)}dU(x)+\int_0^\infty xV_n(x)\overline{F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}$$

$$\overline{U(x)}dF(x)+\int_0^\infty x\overline{F(x)}dU(x)\int_0^\infty V_n(x)\overline{F(x)}dU(x)+\int_0^\infty xV_n(x)$$

$$\overline{F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)+\int_0^\infty x\overline{U(x)}dF(x)\int_0^\infty\overline{F(x)}dU(x)+$$

$$\int_0^\infty x\overline{F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)+\int_0^\infty x\overline{F(x)}dU(x)\int_0^\infty\overline{F(x)}dU(x)+$$

$$\left.\int_0^\infty x\overline{F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)\right)+\int_0^\infty\overline{G_n(x)F(x)}dU(x)$$

$$\int_0^\infty x\overline{U(x)}dF(x)+\int_0^\infty\overline{V_n(x)F(x)}dU(x)\int_0^\infty x\overline{F(x)}dU(x)+$$

$$\int_0^\infty\overline{V_n(x)U(x)}dF(x)\left(\int_0^\infty x\overline{U(x)}dF(x)\int_0^\infty G_n(x)\overline{F(x)}dU(x)+\right.$$

$$\int_0^\infty xG_n(x)\overline{F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)+\int_0^\infty x\overline{F(x)}d$$

$$U(x)\int_0^\infty G_n(x)\overline{F(x)}dU(x)+\int_0^\infty xG_n(x)\overline{F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)+$$

$$\int_0^\infty x\overline{U(x)}dF(x)\int_0^\infty\overline{F(x)}dU(x)+\int_0^\infty x\overline{F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)+$$

$$\int_0^\infty x\overline{F(x)}dU(x)\int_0^\infty\overline{F(x)}dU(x)+\int_0^\infty x\overline{F(x)}dU(x)\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)+$$

[2]We used the inequalities $-(1-e^{-x})\leq 1-e^{-x}$, and $1-e^{-x}\leq x$.

$$\left.\int_0^\infty x\overline{F(x)}dU(x)\right) \leq$$

$$\leq \alpha_n sK\left(\int_0^\infty \overline{G_n(x)U(x)}dF(x) + \int_0^\infty \overline{G_n(x)F(x)}dU(x) + \right.$$

$$\left.\int_0^\infty \overline{V_n(x)F(x)}dU(x) + \int_0^\infty \overline{V_n(x)U(x)}dF(x)\right).$$

Owing to the conditions (2), the expression in the last brackets tends to zero as $n \to \infty$. We have $K \leq 11\max\left(\int_0^\infty \overline{U(x)}dx, \int_0^\infty \overline{F(x)}dx\right)$ which is obvious because in the brackets nearby $\int_0^\infty \overline{G_n(x)U(x)}dF(x)$ we have eleven summands neither of which exceeds $\max\left(\int_0^\infty \overline{U(x)}dx, \int_0^\infty \overline{F(x)}dx\right)$, so that a) is proved. Let us show now that

b) $\lim_{n\to\infty} \frac{{p_n}'(\alpha_n s) - p_n(\alpha_n s)}{\alpha_n}$ uniformly on s on every limited interval.

$${p_n}'(\alpha_n s) - p_n(\alpha_n s) =$$

$$k_1(\alpha_n s)\int_0^\infty (1 - e^{-\alpha_n sx})\overline{G_n(x)U(x)}dF(x) + k_2(\alpha_n s)\int_0^\infty (1 - e^{\alpha_n sx})$$

$$\overline{G_n(x)F(x)}dU(x) + k_3(\alpha_n s)\int_0^\infty (1 - e^{\alpha_n sx})\overline{V_n(x)F(x)}dU(x) +$$

$$k_4(\alpha_n s)\int_0^\infty (1 - e^{-\alpha_n sx})\overline{V_n(x)U(x)}dF(x) \leq$$

$$\leq k_1(\alpha_n s)\alpha_n s\int_0^\infty x\overline{G_n(x)U(x)}dF(x) + k_2(\alpha_n s)\alpha_n s$$

$$\int_0^\infty x\overline{G_n(x)F(x)}dU(x) + k_3(\alpha_n s)\alpha_n s\int_0^\infty x\overline{V_n(x)F(x)}dU +$$

$$k_4(\alpha_n s)\alpha_n s\int_0^\infty x\overline{V_n(x)U(x)}dF(x) \leq$$

$$\leq k_1(\alpha_n s)\alpha_n s\left(A_n^1\int_0^\infty \overline{G_n(x)U(x)}dF(x) + \int_{A_n^1}^\infty x\overline{U(x)}dF(x)\ \right) +$$

$$k_2(\alpha_n s)\alpha_n s\left(A_n^2\int_0^\infty \overline{G_n(x)F(x)}dU(x)+\int_{A_n^2}^\infty x\overline{F(x)}dU(x)\right)+$$
$$k_3(\alpha_n s)\alpha_n s\left(A_n^3\int_0^\infty \overline{V_n(x)F(x)}dU(x)+\int_{A_n^3}^\infty x\overline{F(x)}dU(x)\right)+$$
$$k_4(\alpha_n s)\alpha_n s\left(A_n^4\int_0^\infty \overline{V_n(x)U(x)}dF(x)+\int_{A_n^4}^\infty x\overline{U(x)}dF(x)\right).$$

For $A_n^j =$ (corresponding integral by which we multiply $A_n^j)^{-\frac{1}{2}}$, $j=1,2,3,4$, the items in the brackets tend to zero, and therefore $\lim_{n\to\infty} p_n(\alpha_n s)/\alpha_n = 1$ uniformly on s on every limited interval. The Lemma 1. is proved.

Let us write the denomitaror $p_{1n}(s)$ of the $\mathcal{S}_{rn}(s)$ in the following way:

$$p_{1n}(s)=$$
$$1-\int_0^\infty e^{-sx}\overline{U(x)}dF(x)-\int_0^\infty e^{-sx}\overline{F(x)}dU(x)+\int_0^\infty e^{-sx}\overline{G_n(x)U(x)}dF(x)\Bigg(1-$$
$$\int_0^\infty e^{-sx}\overline{F(x)}dU(x)+\int_0^\infty e^{-sx}\overline{V_n(x)F(x)}dU(x)\Bigg)+$$
$$\int_0^\infty e^{-sx}\overline{V_n(x)F(x)}dU(x)\left(1-\int_0^\infty e^{-sx}\overline{U(x)}dF(x)\right)+$$
$$\int_0^\infty e^{-sx}\overline{G_n(x)F(x)}dU(x)\int_0^\infty e^{-sx}\overline{U(x)}dF(x)=$$
$$=1-\int_0^\infty e^{-sx}\overline{U(x)}dF(x)-\int_0^\infty e^{-sx}dU(x)+q_n(s).$$

Let us notice that the following equalitions are valid $q_n(0)=p_{1n}(0)=\alpha_n$ (because of $d_1(0)+d_2(0)=1$).

LEMMA 2.

$$\lim_{n\to\infty}\frac{q_n(\alpha_n s)}{\alpha_n}=1$$

uniformly on s on every limited interval.

PROOF. Let

$$
\begin{aligned}
&{q_n}'(s) = \\
&\int_0^\infty \overline{G_n(x)U(x)}dF(x)\left(1-\int_0^\infty e^{-sx}\overline{F(x)}dU(x)+\int_0^\infty e^{-sx}\overline{V_n(x)F(x)}dU(x)\right)+\\
&\int_0^\infty \overline{V_n(x)F(x)}dU(x)\left(1-\int_0^\infty e^{-sx}\overline{U(x)}dF(x)\right)+\int_0^\infty \overline{V_n(x)U(x)}dF(x)\\
&\left(\int_0^\infty e^{-sx}\overline{F(x)}dU(x)-\int_0^\infty e^{-sx}\overline{G_n(x)F(x)}dU(x)\right)+\\
&\int_0^\infty \overline{G_n(x)F(x)}dU(x)\int_0^\infty e^{-sx}\overline{U(x)}dF(x).
\end{aligned}
$$

We shall prove that

$$\lim_{n\to\infty}\frac{\alpha_n-{q_n}'(\alpha_n s)+{q_n}'(\alpha_n s)-q_n(\alpha_n s)}{\alpha_n}=0$$

uniformly on s on every limited interval. Really

$$
\begin{aligned}
&\alpha_n-{q_n}'(\alpha_n s)+{q_n}'(\alpha_n s)-q_n(\alpha_n s)=\int_0^\infty \overline{G_n(x)U(x)}dF(x)\left(-\int_0^\infty(1-e^{-\alpha_n sx})\right.\\
&\left.\overline{F(x)}dU(x)+\int_0^\infty(1-e^{-\alpha_n sx})\overline{V_n(x)F(x)}dU(x)\right)+\int_0^\infty \overline{V_n(x)F(x)}dU(x)\left(-\int_0^\infty(1-\right.\\
&\left.e^{-\alpha_n sx})\overline{U(x)}dF(x)\right)+\int_0^\infty \overline{V_n(x)U(x)}dF(x)\left(\int_0^\infty(1-e^{-\alpha_n sx})\ \overline{F(x)}dU(x)-\int_0^\infty(1-\right.\\
&\left.e^{-\alpha_n sx})\overline{G_n(x)F(x)}dU(x)\ \right)+\int_0^\infty \overline{G_n(x)F(x)}dU(x)\int_0^\infty(1-e^{-\alpha_n sx})\overline{U(x)}dF(x)+\int_0^\infty(1-\\
&e^{-\alpha_n sx})\overline{G_n(x)U(x)}dF(x)\left(1-\int_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)+\int_0^\infty e^{-\alpha_n sx}\overline{V_n(x)F(x)}dU(x)\right)+\\
&\int_0^\infty(1-e^{-\alpha_n sx})\overline{V_n(x)F(x)}dU(x)\left(1-\int_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x)\right)+\int_0^\infty(1-e^{-\alpha_n sx})\overline{V_n(x)}
\end{aligned}
$$

$$\overline{U(x)}dF(x)\left(\int\limits_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x) - \int\limits_0^\infty e^{-\alpha_n sx}\overline{G_n(x)F(x)}dU(x) + \right.$$
$$\int\limits_0^\infty \overline{G_n(x)F(x)}dU(x)\int\limits_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x) \leq \alpha_n o(1).$$

The last inequality is obtained in the same way as the analogous inequality in Lemma 1., and therefore we are not going to repair it.

Let us come back to the proof of the Theorem 1.

By τ we denote the random variable which corresponding to the time without failure of the two-unit system; let $\Phi(x)$ be the distribution function of τ.

Then $P\{\alpha_n\tau < t\} = \Phi(t/\alpha_n)$ and $\int_0^\infty e^{-st}d\Phi(t/\alpha_n) = \mathcal{S}_{rn}(\alpha_n s)$. On the other hand

$$\mathcal{S}_{rn}(\alpha_n s) = \frac{p_n(\alpha_n s)}{1 - \int\limits_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x) - \int\limits_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x) + q_n(\alpha_n s)} =$$
$$= \frac{\frac{p_n(\alpha_n s)}{\alpha_n}}{\frac{1-\int\limits_0^\infty e^{-\alpha_n sx}\overline{U(x)}dF(x) - \int\limits_0^\infty e^{-\alpha_n sx}\overline{F(x)}dU(x)}{\alpha_n} + \frac{q_n(\alpha_n s)}{\alpha_n}} \underset{n\to\infty}{\longrightarrow} \frac{1}{1+sM}$$

where $M = \int_0^\infty x\overline{U(x)}dF(x) + \int_0^\infty \overline{F(x)}dU(x)$, i.e. the limiting distribution function is exponential:

$$\lim_{n\to\infty} P\{\alpha_n\tau < t\} = 1 - \exp(-t/M).$$

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