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ON THE FUNCTIONAL EQUATION $f\varphi f = f$

Valentina Harizanov

Abstract. In this note we determine the general solution of the equation $f\varphi f = f$, where $f: X \to Y$ is a given function and $\varphi: Y \to X$ is an unkown function (X and Y are arbitrary nonempty sets). The general solution of that equation is given by the formula (4), where $\varphi_0: Y \to X$ is a particular solution, $k: Y \to X$ and $h: X \to X$ are arbitrary functions, $F: X^3 \times Y^3 \to X$ is defined by (3).

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Let X and Y be nonempty sets and f a given function from X to Y. By a generalized inverse of the function f we mean every function φ from Y to X which is a solution of the functional eguation

$$f\varphi f = f, \tag{1}$$

i.e. for every $x \in X$, $f(\varphi(f(x))) = f(x)$. The condition that the equation (1) has a solution is equivalent to the axiom of choise, as can be easily shown. In the case that f is a bijection there exists the unique solution of (1) and it is the inverse function of f (defined as usual). The following theorem describes (in a certain way) all the solutions of the functional equation (1), provided that its particular solution is known. We reason in the following way:

Let $f: X \to Y$ be any function. Then the relation $\sim$ on X, defined by $x \sim y \Leftrightarrow f(x) = f(y)$, is an equivalence relation and the corresponding quotient set is $X/\sim = \{C_y \mid y \in f(X)\}$, where $C_y = f^{-1}(y)$. A function $\varphi: Y \to X$ is a solution of the equation (1) if and only if the following condition is satisfied

$$(\forall y \in f(X))(\varphi(y) \in C_y). \tag{2}$$

This implies that for $y \in Y \backslash f(X)$, $\varphi(y)$ can be arbitrarily chosen. In order to fulfill the condition (2) we shall use, beside a particular solution φ_0 of the equation, an arbitrary function h from X to X.

In the construction of the formula which gives the general solution of the equation (1) we shall also use the function $F: X^3 \times Y^3 \to X$, defined by

$$(3) \qquad F(x,y,z;u,v,w) = \begin{cases} x, & \text{if } u \neq w, \\ y, & \text{if } u = w \text{ and } u \neq v, \\ z, & \text{if } u = v = w, \end{cases}$$

where $x,\ y,\ z \in X$ and $u,\ v,\ w \in Y$. Since the conditions on the right-hand side exclude each other and form a complete system, F is well-defined[1].

Theorem. *If $\varphi_0: Y \to X$ is a particular solution of the functional equation* (1)*, then its general solution is given by*

$$(4) \qquad \varphi(x) = F(k(x), \varphi_0(x), h(\varphi_0(x)); f(\varphi_0(x)), f(h(\varphi_0(x))), x) \qquad (x \in Y),$$

where $F: X^3 \times Y^3 \to X$ is a function defined by (3) *and $k: Y \to X$, $h: X \to X$ are arbitrary functions.*

Proof. Let $k: Y \to X$ and $h: X \to X$ be arbitrary functions. Then for φ defined by (4) and for every $x \in X$ we have[2]

$$\begin{aligned} \varphi f x &= F(kfx, \varphi_0 fx, h\varphi_0 fx; f\varphi_0 fx, fx, fh\varphi_0 fx, fx) \\ &= \begin{cases} kfx, & \text{if } f\varphi_0 fx \neq fx, \\ \varphi_0 fx, & \text{if } f\varphi_0 fx = fx \text{ and } f\varphi_0 fx \neq fh\varphi_0 fx, \\ h\varphi_0 fx & \text{if } f\varphi_0 fx = fh\varphi_0 fx = fx. \end{cases} \end{aligned}$$

Since $f\varphi_0 fx = fx$, we get

$$\varphi fx = \begin{cases} \varphi_0 fx, & \text{if } fx \neq fh\varphi_0 fx, \\ h\varphi_0 fx & \text{if } fx = fh\varphi_0 fx. \end{cases}$$

Finally,

$$\begin{aligned} f\varphi_0 fx &= \begin{cases} f\varphi_0 fx, & \text{if } fx \neq fh\varphi_0 fx, \\ fh\varphi_0 fx, & \text{if } fx = fh\varphi_0 fx \end{cases} \\ &= \begin{cases} fx, & \text{if } fx \neq fh\varphi_0 fx, \\ fx, & \text{if } fx = fh\varphi_0 fx \end{cases} \\ &= fx, \end{aligned}$$

i.e. φ satisfies the equation (1).

Coversely, let $\varphi: Y \to X$ be a solution of (1). We shall show that φ can be written in the form (4). Let $k: Y \to X$ be equal to φ and $h: X \to X$ be defined by

$$hy = \begin{cases} \varphi x, & \text{if } \varphi_0 x = y \text{ and } f\varphi_0 x = x \\ & \text{for some } x \in Y, \\ \text{arbitrary}, & \text{otherwise}, \end{cases}$$

[1] We can call the function F a resolution function.

[2] For the sake of simplicity we shal write kh, $h\varphi_0 x$ etc. instead of $k(x)$, $h\varphi_0(x))$, ...

where $y \in X$. The function h is well-defined, since hy does not depend on the choise of x. Indeed, assuming that there exist, x, $x' \in Y$ such that $\varphi_0 x = y$, $\varphi_0 x' = y$, $f\varphi_0 x = x$, $f\varphi_0 x' = x'$, we get $x = fy = x'$.

Then for functions k and h and $x \in Y$ we get

$$
\begin{aligned}
&F(kx, \varphi_0 x, h\varphi_0 x; f\varphi_0 x, fh\varphi_0 x, x)\\
&= \begin{cases} \varphi x, & \text{if } f\varphi_0 x \neq x,\\ \varphi_0 x, & \text{if } \varphi_0 x = x \text{ and } f\varphi_0 x \neq fh\varphi_0 x,\\ h\varphi_0 x, & \text{if } f\varphi_0 x = fh\varphi_0 x = x \end{cases}
\end{aligned}
$$

(by $k = \varphi$)

$$
= \begin{cases} \varphi x, & f\varphi_0 x \neq x,\\ \varphi_0 x, & \text{if } f\varphi_0 x = x \text{ and } f\varphi_0 x \neq f\varphi x,\\ \varphi x, & \text{if } f\varphi_0 x = f\varphi x = x \end{cases}
$$

(Applying the definition of h, from $f\varphi_0 x = x$ we obtain $hy = \varphi x$ for $y = \varphi_0 x$, i.e. $h\varphi_0 x = \varphi x$.)

$$
\begin{cases} \varphi x, & \text{if } f\varphi_0 x \neq x,\\ \varphi x, & \text{if } f\varphi_0 x = x \end{cases}
$$

(From $f\varphi_0 x = x$ and $f\varphi f = f$ it follows $f\varphi x = f\varphi f\varphi_0 x = f\varphi_0 x$, which contradicts $f\varphi x \neq f\varphi_0 x$.)

$$= \varphi x.$$

This proves the theorem.

In connection with the previous theorem we observe that if the function f is surjective, then $f\varphi_0 x = x$ for every $x \in Y$. In that case only one arbitrary function ($h\colon X \to X$) occurs in the formula for the general solution of the equation (1):

$$
\begin{aligned}
\varphi x &= F(kx, \varphi_0 x, h\varphi_0 x; x, fh\varphi_0 x, x)\\
&\begin{cases} \varphi_0 x, & \text{if } fh\varphi_0 x \neq x,\\ h\varphi_0 x, & \text{if } fh\varphi_0 x = x. \end{cases}
\end{aligned}
$$

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Valentina Harizanov
11000 Beograd
Ustanička 141