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Q_r-SEMIGROUPS

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T. E. Nordahl, in [5], considered the commutative Q-semigroups. C. S. H. Nagore, [4] extended Nordahl's results on quasi-commutative semigroups. A. Cherubini-Spoletini and A. Varisco consider Putcha's Q-semigroups. The definition of weakly commutative semigroup has given by M. Petrich in [6]. Here, we give the definition of Q_r-semigroup i.e. a semigroup in which every proper right ideal is a power joined semigroup and we give as well some characterizations of weakly commutative Q_r-semigroups, (Theorem 3.1.).

In section 1. we characterize semilattices of groups. In section 2. we consider archimedean weakly commutative semigroups. A weakly commutative semigroup which does not have prime ideals is characterized by Theorem 2.1. This theorem is a generalization of G. Thierrin's result, [11]. By Theorem 2.3. are characterized weakly commutative semigroups with an idemotent which are archimedean. This theorem is an extension of T. Tamura and N. Kimura's result in [10]. G. Thierrin and G. Thomas in [12], too, give a characterization for these semigroups. In section 3. we give the definition of Q_r-semigroup. This notion is another generalization of the notion of power joined semigroup. The description of weakly commutative Q_r-semigroups is given by Theorem 3.1.

For undefined notions we reffer to [2] and [7].

1. Semilattices of groups

Here, we will characterize the semilattices of groups using the notion of weakly commutative semigroup.

Definition 1.1. [6]. A semigroup S is weakly commutative if for every a, $b \in S$ there exist x, $y \in S$ and $n \in N$ such that

$$(ab)^n = xa = by. \tag{1.1}$$

Denote with π the class of all weakly commutative semigroups.

THEOREM 1.1. *Let S be a semigroup. Then S is a semilattice of groups if and only if $S \in \pi$ and S is a (left, right, intra-) regular.*

PROOF. Let S be a regular semigroup. Then for every $a \in S$ there exists $x \in S$ such that $a = axa$. Hence, $a = (ax)^n a$, for every $n \in N$. As $S \in \pi$, then for a and x there exist $m \in N$ and $z \in S$ such that $(ax)^m = za$, so $a = (ax)^m a = za^2 \in Sa^2$. Hence, S is a left regular semigroup. Similarly we have that S is a right regular semigroup. By Theorem 12. [8] we have that S is a semilattice of groups.

The converse follows by Theorem 12. [8].

COROLLARY 1.1 *Let S be a semigroup. Then S is a (left, right) simple and $S \in \pi$ if and only if S is a group.*

2. Archimedean semigroups

DEFINITION 2.1. [10]. A semigroup S is left (right) arhimedean is for every $a, b \in S$ there exist $x, y \in S$ and $n \in N$ such that $a^n = xb$, $b^n = ya$, $(a^n = bx, b^n = ay)$. S is an archimedean semigroup if for every $a, b \in S$ there exist $x, u, y, v \in S$ and $n \in N$ such that $a^n = xby$, $b^n = uav$.

LEMMA 2.1. *Let $S \in \pi$. Then, the following conditions are equivalent:*

(i) *S is left archimedean,*

(ii) *S is right archimedean,*

(iii) *S is archimedean.*

PROOF. (i) $\Rightarrow$ (ii). Let for every $a, b \in S$ exist $x, y \in S$ and $n \in N$ such that

$$a^n = xb, \quad b^n = ya. \tag{2.1}$$

As $S \in \pi$, then for x and b there exist $m \in N$ and $z, u \in S$ such that

$$(xb)^m = bz = ux. \tag{2.2}$$

Similarly

$$(ya)^k = av = wy \tag{2.3}$$

for some $k \in N$ and $v, w \in S$. From (2.1) and (2.2) we have that

$$a^{nm} = (xb)^m = bz. \tag{2.4}$$

From (2.1) and (2.3) we have

$$b^{nk} = (ya)^k = av. \tag{2.5}$$

From (2.4) and (2.5) it follows

$$a^{nmk} = (bz)^k, \ \ b^{nmk} = (av)^m.$$

Hence, S is a right archimedean semigroup. Similarly to the previous, it can be proved that $(iii) \Rightarrow (i)$. $(ii) \Rightarrow (iii)$ follows immediately.

Corollary 2.1. *A weakly commutative archimedean semigroup has one idempotent at most.*

Lemma 2.2. *Let S be a weakly commutative archimedean semigroupt. Then every semiprime ideal from S is two-sided.*

Proof. Let $S \in \pi$ and R be a right ideal of S and R is semiprime. For arbitrary $a \in R$, $b \in S$ there exist x, $y \in S$ and $n \in N$ such that $(ba)^n = ax \in R$, hence $ba \in R$. Similarly, for a left ideal of S.

Theorem 2.1. *Let S be a semigroup. Then S is weakly commutative and S does not have proper prime ideals if and only if S is a left and right archimedean semigroup.*

Proof. Let S be a weakly commutative semigroup that does not have proper prime ideals. Let $\langle a \rangle$ be a cyclic semigroup generated by $a \in S$. Denote with S_a the set of all $x \in S$ such that they divide from the left side some element from $\langle a \rangle$. The set S_a is non-empty since $\langle a \rangle \subset S_a$. The set S_a a subsemigroup of S. For x, $y \in S_a$ exist u, $w \in S^1$ and $h \in N$ such that $ux = a^h$, $wy = a^h$ and exist $v \in S^1$ and $k \in N$ such that $yv = a^k$, (Lemma 2.1.), so $u(xy)v = a^{h+k}$, (Lemma 2.1.). Hence, $xy \in S_a$. Take $S \backslash S_a \neq \emptyset$ and $z \in S \backslash S_a$, $a \in S$. The element az is not in S_a, (if $az \in S_a$, then there exist $u \in S$ such that $uaz \in \langle a \rangle$, so $z \in S_a$, which is impossible). Hence, $az \in S \backslash S_a$, so $S \backslash S_a$ is a left ideal of S. Since S_a is a subsemigroup of S, so $S \backslash S_a$ is a prime ideal of S, hence it is two-sided, (Lemma 2.2.). Let a, $b \in S$. As S does not have proper prime ideals it follows that $S \backslash S_a = \emptyset$, i.e. $S = S_a$ and there exist $u \in S^1$ and $h \in N$ such that $a^h = ub$. Analogously $b^k = va$, $(k \in N, v \in S^1)$. Hence, S is a left archimedean semigroup. If can be proved, in a similar way, that S is a right archimedean semigroup.

Conversely, let S be a left and right archimedean semigroup. Then S is weakly commutative. Let S has a proper prime ideal I and let $a \in I$, $b \in S \backslash I$. Then there exist $x \in S$ and $n \in N$ such that $b^n = ax \in I$, so $b \in I$, which is impossible.

Lemma 2.3. *Let $S \in \pi$ be a archimedean semigroup with the idempotent e, then eS is a group and $eS = Sd = SeS$ hold.*

Proof. Let $a \in eS$. Then $a = ex$ for some $x \in S$. From this we have $ea = e^2x = ex = a$, so e is a left identity for eS. S is an archimedean semigroup, then it exists $y \in S$ such that $e = ya$, (Lemma 2.1.) i.e. $e = (ey)a$. Hence, a has in eS an inverse element relatively to e. It follows that eS is a group with identity e. For arbitrary $a \in eS$, $a = ex$ holds, $x \in S$, so $a = eex \in SeS$. Similarly for arbitrary $b \in SeS$ is $b = uev$, $(u, v \in S)$, i.e. $b = u(ev)e = (uev)e \in Se$, because e is an identity in eS. Hence,

$$eS \subset SeS \subset Se. \tag{2.6}$$

We prove that

$$Se \subset SeS \subset eS \tag{2.7}$$

analogously. From (2.6) and (2.7) we have that $eS = Se = SeS$.

By using the Theorem of Clifford (Theorem 4; 19. [2]) it can be easily verified.

Lemma 2.4. *If S is an ideal extension of a weakly commutative archimedean semigroup with identity by a nil-semigroup, then S is weakly commutative semigroup.*

The following theorem is an extension of the result of T. Tamura and N. Kimura [10].

Theorem 2.3. *Let S be a semigroup. Then S is a weakly commutative archimedean with an idempotent if and only if S is a group or S is an ideal extension of a group by a nil-semigroup.*

Proof. Let S be a weakly commutative archimedean semigroup with the idempotent e. If S is simple, then S is a group (Corollary 1.1.). If S is not simple, take the ideal $I = SeS$ and the factor-semigroup of Rees S/I. From Lemma 2.3. I is a group. Since S is an archimedean semigroup, so for every $a \in S$, $b \in I$ there exist a natural number n and $x \in S$ such that $a^n = bx$ holds, (Lemma 2.1.). From this we have that $a^n \in I$. Hence, S/I is a nil-semigroup. If e is a zero in S then $S/I \cong S$, so S is a nil-esmigroup itself, because I contains only e.

Conversely, let S be an extension of the group I by a nil-semigroup Q. From Lemma 2.4. S is a weakly commutative semigroup. Obviously, S contains only one idempotent (an identity from I). Let us prove that S is an archimedean semigroup. The semigroup S/I is a nil-semigroup, and, as $S/I \cong Q$, it follows that for arbitrary a, $b \in S$ there exist h and k such that a^h, $b^k \in I$. But, I is a group, so there exist x, $y \in I$ such that $a^h = b^k x$, $b^k = a^h y$. Hence, S is a right archimedean semigroup, so from Lemma 2.1. it is archimedean. The assertion follows immediality if S is a group.

Lemma 2.5. *Let S be an archimedean weakly comutative semigroup without idempotents. Then $a \neq ab$, for every a, $b \in S$.*

Proof. Let S be an archimedean weakly commutative semigroup without idempotents. Assume opposite, i.e. $a = ab$. Then for a and b there exist $x \in S$ and $n \in N$ such that $b^n = ax$ holds, (Lemma 2.1.), and so $a = ab = ab^2 = \cdots = ab^n$, so $a = a^2x$. Hence, the element a is a right regular, so it is regular, (Theorem 1.1.). It follows that S has an idempotent, which is impossible.

3. Q_r-Semigroups

Definition 3.1. [6]. A semigroup S is a power joined if for every a, $b \in S$ there exist m, $n \in N$ such that $a^m = b^n$.

Obviously, a power joined semigroup is weakly commutative.

Immediately follows

Lemma 3.1. *Let S be a semigroup. Then the following conditions are equivalent:*

(i) *S is power joined,*

(ii) *Every ideal from S is a power joined semigroup,*

(iii) *Every right (left) ideal of S is a power joined semigroup.*

T. E. NORDAHL, [5] considered commutative Q-semigroups. We give here the definition of Q_r-semigroup, which is another generalization of a power joined semigroup.

Definition 3.2. A semigroup S is Q_r-semigroup (Q_r-semigroup) if every proper right (left) ideal of S is a power joined semigroup.

Q_r-semigroup is Q-semigroup. The converse is not true. For example, the semigroup S given by

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & a & a & a & a \\ b & a & a & a & b \\ c & a & a & a & c \\ d & a & a & a & d \end{array}$$

is a Q-semigroup. But, the right ideal $\{a, d\}$ is not a power joined semigroup, so S is not a Q_r-semigroup.

The following theorem describes weakly commutative Q_r-semigroups.

Theorem 3.1. *Let S be a semigroup. Then S is a weakly commutative Q_r-semigroup if and only if one of the tree possibilites hold:*

1° *S is a power joined semigroup,*

1° *S is a group,*

3° *$S = M \cup G$ and the identity e of the group G is a left identity of S and M is the unique maximal prime ideal of S and M is a power joined semigroup.*

Proof. Let S be an archimedean weakly commutative Q_r-semigroup. Then S has one idempotent at most, (Corollary 2.1.). If S does not have an idempotent, then from Lemma 2.5. for every $a \in S$ is $a \notin aS$. From this, we have that aS is a proper right ideal of S. Hence, aS is a power joined semigroup. For $b \in S$ there exists $n \in N$ such that $b^n \in aS$, (Lemma 2.1) and obviously there exists

$m \in N$ such that $a^m \in aS$. aS is a power joined semigroup, then there exist r, $s \in N$ such that $a^{mr} = b^{ns}$. Hence, in this case S is a power joined semigroup. If S has an idempotent e, then from Lemma 2.3. eS is a group-ideal of S. If $eS \neq S$, then eS is a proper ideal of S, eS is power joined, so eS is a periodic group. So, S is a nil-extension of a periodic group, (Theorem 2.3.). From this eS is a power joined semigroup with one idempotent. If $es = S$, then S is a group. If S is not an archimedean semigroup then from Theorem 2.1. S has a proper prime ideal. Denote with M the union of all proper prime ideals of S. Then M is a maximal prime ideal of S and M is a power joined semigroup. If $M = S$, then S is a power joined semigroup. If $M \not\subset S$, then for $x \in S \backslash M$ is $x^2 \in S \backslash M$ and as M is a maximal ideal of S, $M \cup J(x) = M \cup J(x^2) = S$, so $x = x^2$ or $x = x^2 t$ or $x = t_1 x^2$ or $x = t_2 x^2 t_3$, for some t, t_1, t_2, $t_3 \in S \backslash M$. From the Theorem 1.1 we have that in each of these cases x is a regular element, i.e. $S \backslash M$ is a regular semigroup, so it contains idempotents. It can be easily verified that $S \backslash M$ has only one idempotent. Hence, $S \backslash M$ is a group. So if $M \not\subseteq S$, then

$$S = M \cup G \tag{*}$$

where M is a unique maximal prime ideal which is a power joined semigroup and G is a group. We distinguish now two cases:

(i) $eS = S$. Then for each $x \in S$ is $x = es$, for som $z \in S$ and $ex = e(es) = x$. Hence, e is a left identity of S, aod in this case S is of the type 3°.

(ii) $eS \not\subseteq S$. Then eS is a power joined semigroup. From (*) we have that

$$eS = eM \cup eG = eM \cup G.$$

It follows that $G \subset eS$, so $S = M \cup eS$ which means that in this case S is a power joined semigroup.

Conversely, let 3° holds. If $a, b \in M$ then there exist $x, y \in M$ and $n \in N$ such that $(ab)^n = xa = by$, because M is power joined. If a, $b \in G$ then $(ab)^n = xa = by$ for some x, $y \in G$ and $n \in N$. If $a \in M$, $b \in G$, then $bab \in M$, so

$$(ab)^{2k} = [a(bab)]^k = xa = baby, \text{ for some } k \in N \text{ and some } x, y \in M.$$

Hence, S is a weakly commutative semigroup. Take an arbitrary proper right ideal R from S. If $R \subset M$, then R is a power joined semigroup, so S is a Q_r-semigroup. If $R \cap G \neq \emptyset$, then $G \subset R$ and we have that $e \in R$. Hence, $R = S$, which is impossible. In the other cases the assertion immediately follows.

Note that Lemma 3.1. holds if we change the term "ideal" with the term "quasi-ideal" ("bi-ideal"), (for definitions of a quasi-ideal and bi-ideal see [2] or [9]). Hence, the notion of a power joined semigroup could be generalised in the following way:

Definition 3.3. A semigroup S is a Q_q-semigroup (Q_b-semigroup) if every proper quasi-ideal (bi-ideal) of S is a power joined semigroup.

Denote with P, Q_b, Q_r, Q_l, Q_q, Q the classes of all power joined, Q_b-, Q_r-, Q_l-, Q_q-, Q-semigroups. The we have

LEMMA 3.2. $P \subset Q_b \subset Q_q \subset Q_r \cup Q_l \subset Q$.

From the Theorem 3.1. its dual theorem and lemma 3.2. immediately follows

THEOREM 3.2. *Let S be a weakly commutative archimedean semigroup with no idempotents, then the following conditions are equivalent:*

(i) *S is power joined,*

(ii) *S is Q_b-semigroup,*

(iii) *S is Q_r-semigroup,*

(iv) *S is Q_l-semigroup,*

(v) *S is Q_q-semigroup.*

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