PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 28 (42), 1980, pp. 209–213

ON THE RECONSTRUCTION OF LATIN SQUARES

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Abstract. We consider the following problem: Find the least integer $N(n)$ such that for arbitrary latin square L of order n we can choose $N(n)$ cells of that square such that after erasing the enteries occupying the remaining $n^2 - N(n)$ cells the latin square L can be reconstrusted uniquely. We discuss in detail the cases $n \leq 6$.

A *latin rectangle* of order r by s, where $r \leq s$ ($s \leq r$), is an array of r rows and s columns formed from s symbols (r symbols) such that each row (column) contains all the symbols, and no column (row) contains any symbol more that once. If $r = s = n$ then latin rectangle is called *latin square* of order n. We shall consider latin squares based upon integers $1, 2, \dots, n$. *Partial (or incomplete) latin square* of order n is an n by n array such that in some subset of the n^2 cells of array each of the cells is occupied by an integer from the set $\{1, 2, \dots, n\}$ and that no integer from that set occurs in any row or column more than once.

Now, we give some new definitions.

Definitions 1. Let L_p be a partial latin square of order n which can be embedded in the latin square L of order n and can not be embedded in any latin square of order n different from L. If no one partial latin square obtained from L_p by erasing some of its entries has not this property we say that L_p is a *skeleton* of L.

Definition 2. A skeleton L_p of the latin square L is said to be *minimal* iff the number of entries of L_p (the number of occupied cells) is not greater than the number of entries of any other skeleton of L.

It is clear that a latin square can have more than one minimal skeleton. Let k_L denotes the number of entries of a minimal skeleton of the latin square L. Then the following statement holds:

Theorem: *If the latin squares L and L' are isotopic, then $k_L = k_{L'}$.*

Proof. Enclose in squares k_L entries of the latin square L belonging to a skeleton L_p of L. Isotopy of L and L' implies that L can be transformed into

L' by rearranging rows, rearranging columns and renaming elements. The number of enclosed entries after these manipulations rest unchanged. If the partial latin square ${L_p}'$ consisting from enclosed entries of L' is not a skeleton of L', then ${L_p}'$ can also be embedded in some latin square ${L_1}'$ of order n diferent from L'. Now, by the same rearranging rows, rearranging columns and by inverse renaming elements, ${L_1}'$ will be transformed into the latin square L_1 of order n different from L such that L_p is embedded in L_1. It is a contradiction because, by hypothesis, L_p is a skeleton of L. So, to any skeleton of L corresponds examply one skeleton with the same number of entries of the isotopic latin square L', and conversely. Hence follows the statement.

LEMMA 1. Let the latin rectangle P of order r by $s(1 < r, s \leq n)$ based upon integers $a_1, a_2, \dots, a_p (p = \max(r, s))$ from the set $\{1, 2, \dots, n\}$ is a subrectangle of the latin square L of order n based upon integers $1, 2, \dots, n$. Then any skeleton of L contains at least one entry from P.

PROOF. Suppose that $r \leq s$ (the case $s \leq r$ is analog). Then any row of P contains each of s elements. The latin square L can be transformed in a latin square L' different from L, by rearringing rows of P. Any partial latin square L_p of order n embedded in L and not containing at least one entry from P can not be a skeleton because L_p is embedded in L' too.

LEMMA 2. Let the latin square L' of order $r < n$ be a subsquare of the latin square L of order n. Then any skeleton of L contains at least $k_{L'}$ entries from L'.

PROOF. The entries belonging at the same time to the subsquare L' and to the skeleton L_p of the latin square L from the skeleton ${L_p}'$ of L'. For, if it is not true then the partial latin square ${L_p}'$ can be embedded in at least two latin squares of order r, and consequently, the partial latin square L_p can be embedded in at least two latin squares of order n, but it is contradiction.

LEMMA 3. For arbitrary latin square L of order n, the inequalities

$$n - 1 \leq k_L \leq (n-1)^2$$

hold.

PROOF. The first inequality holds because, according to Lemma 1, there exists at most one row of the latin square L not contraining at least one entry belonging to the arbitrary skeleton of L.

The second inequality holds because the arbitrary latin square of order n can be reconstructed from $(n-1)^2$ entries not belonging to the n-th row nor to the n-th column.

THEOREM 2. *Let S_n denotes the set of all latin squares of order n and let*

$$N(n) = \max_{L \in S_n} k_L.$$

Then:
a) $N(2) = 1$
b) $N(3) = 2$
c) $N(4) \leq 5$
d) $N(5) \leq 7$
e) $N(6) \leq 13$.

Proof. According to Theorem 1, it is sufficient, for $n = 2, 3, 4, 5, 6$, to consider one representative from each isotopy class of S_n and show that for such representative there exists a partial latin square of the same order, with at most $N(n)$ entries, from which that representative can be reconstructed uniquely. In the process of effective search for such partial latin square we use Lemmas 1, 2, and 3. In the list of squares given below, the entries belonging to such partial latin squares are enclosed in squares.

It is known that there are one isotopy class in S_2, one isotopy class in S_3, two isotopy classes in S_4, two isotopy classes in S_5 and 22 isotopy classes in S_6 (see [1], pp. 128–137). For $n = 6$ it is sufficient to consider only 17 isotopy classes because the squares of the remaining five isotopy classes can be obtained from the squares of some other classes by reflection in their main left-to-right diagonals.

We complete the proof by the following list (the numeration of squares is from [1], pp. 129–137, where the full information concerning classification of latin squares can be found):

a) $n = 2$

$$\begin{array}{|cc|}\hline \boxed{1} & 2 \\ 2 & 1 \\ \hline\end{array}$$

1.1.1.

b) $n = 3$

$$\begin{array}{|ccc|}\hline \boxed{1} & 2 & 3 \\ 2 & \boxed{3} & 1 \\ 3 & 1 & 2 \\ \hline\end{array}$$

1.1.1.

c) $n = 4$

$$\begin{array}{|cccc|}\hline \boxed{1} & 2 & 3 & 4 \\ 2 & 1 & \boxed{4} & 3 \\ \boxed{3} & 4 & 1 & \boxed{2} \\ 4 & \boxed{3} & 2 & 1 \\ \hline\end{array}$$

1.1.1.

$$\begin{array}{|cccc|}\hline \boxed{1} & 2 & 3 & 4 \\ \boxed{2} & 3 & 4 & \boxed{1} \\ 3 & \boxed{4} & 1 & 2 \\ 4 & 1 & 2 & 3 \\ \hline\end{array}$$

2.1.1.

d) $n = 5$

$$\begin{array}{|ccccc|}\hline \boxed{1} & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & \boxed{2} \\ 4 & 5 & 1 & \boxed{2} & \boxed{3} \\ \boxed{5} & \boxed{1} & 2 & 3 & 4 \\ \hline\end{array}$$

1.1.1.

$$\begin{array}{|ccccc|}\hline \boxed{1} & 2 & \boxed{3} & 4 & 5 \\ \boxed{2} & 1 & 4 & 5 & 3 \\ 3 & \boxed{5} & 1 & 2 & \boxed{4} \\ 4 & 3 & 5 & 1 & 2 \\ 5 & \boxed{4} & 2 & 3 & \boxed{1} \\ \hline\end{array}$$

2.1.1.

e) $n = 6$

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	4	5	6
2	3	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	1
$\boxed{3}$	4	5	6	1	2
4	5	6	1	2	3
$\boxed{5}$	6	1	2	3	$\boxed{4}$
6	$\boxed{1}$	$\boxed{2}$	3	$\boxed{4}$	5

1.1.1.

$\boxed{1}$	2	3	4	$\boxed{5}$	6
2	1	5	$\boxed{6}$	$\boxed{3}$	4
3	$\boxed{6}$	1	$\boxed{5}$	4	2
$\boxed{4}$	5	6	1	2	$\boxed{3}$
$\boxed{5}$	$\boxed{4}$	$\boxed{2}$	3	6	1
6	3	$\boxed{4}$	$\boxed{2}$	1	5

2.1.1.

$\boxed{1}$	2	$\boxed{3}$	$\boxed{4}$	5	6
2	$\boxed{3}$	1	5	$\boxed{6}$	4
3	1	2	6	4	5
$\boxed{4}$	6	5	$\boxed{2}$	1	3
$\boxed{5}$	4	6	3	$\boxed{2}$	1
6	$\boxed{5}$	4	1	$\boxed{3}$	2

3.1.1.

$\boxed{1}$	$\boxed{2}$	3	4	5	6
2	1	$\boxed{4}$	3	$\boxed{6}$	5
$\boxed{3}$	4	5	6	1	2
4	3	6	$\boxed{5}$	2	$\boxed{1}$
$\boxed{5}$	6	1	$\boxed{2}$	4	$\boxed{3}$
6	5	2	1	$\boxed{3}$	$\boxed{4}$

4.1.1.

1	2	$\boxed{3}$	4	5	$\boxed{6}$
$\boxed{2}$	1	4	$\boxed{3}$	6	$\boxed{5}$
$\boxed{3}$	$\boxed{4}$	5	6	1	2
4	3	6	$\boxed{5}$	$\boxed{2}$	1
5	$\boxed{6}$	2	1	$\boxed{4}$	3
6	5	$\boxed{1}$	$\boxed{2}$	3	4

5.1.1.

$\boxed{1}$	2	3	4	5	6
2	1	$\boxed{4}$	$\boxed{5}$	$\boxed{6}$	3
$\boxed{3}$	4	$\boxed{5}$	6	2	1
4	$\boxed{6}$	1	2	3	$\boxed{5}$
5	$\boxed{3}$	6	$\boxed{1}$	4	$\boxed{2}$
$\boxed{6}$	5	2	3	$\boxed{1}$	4

6.1.4.

$\boxed{1}$	2	$\boxed{3}$	4	5	6
2	1	$\boxed{4}$	3	6	$\boxed{5}$
$\boxed{3}$	5	$\boxed{1}$	6	4	$\boxed{2}$
4	6	5	1	$\boxed{2}$	3
5	3	6	$\boxed{2}$	1	$\boxed{4}$
$\boxed{6}$	$\boxed{4}$	2	5	$\boxed{3}$	1

7.1.1.

$\boxed{1}$	2	3	$\boxed{4}$	$\boxed{5}$	6
2	1	6	5	$\boxed{3}$	$\boxed{4}$
3	6	$\boxed{1}$	2	4	5
4	5	2	$\boxed{1}$	6	$\boxed{3}$
$\boxed{5}$	3	4	$\boxed{6}$	$\boxed{1}$	2
$\boxed{6}$	4	$\boxed{5}$	3	2	1

8.1.1.

$\boxed{1}$	2	$\boxed{3}$	4	$\boxed{5}$	6
2	1	5	$\boxed{6}$	$\boxed{4}$	3
$\boxed{3}$	4	$\boxed{1}$	$\boxed{5}$	6	$\boxed{2}$
4	3	6	1	$\boxed{2}$	5
$\boxed{5}$	6	2	3	1	4
6	5	$\boxed{4}$	2	3	$\boxed{1}$

8.2.1.

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	4	5	6
2	3	1	$\boxed{6}$	$\boxed{4}$	5
$\boxed{3}$	1	2	5	6	$\boxed{4}$
4	$\boxed{5}$	6	$\boxed{1}$	3	2
$\boxed{5}$	6	4	3	$\boxed{2}$	1
6	$\boxed{4}$	$\boxed{5}$	2	1	3

9.2.1.

$\boxed{1}$	2	3	4	5	$\boxed{6}$
$\boxed{2}$	$\boxed{3}$	1	6	4	5
$\boxed{3}$	$\boxed{1}$	2	$\boxed{5}$	6	4
4	6	$\boxed{5}$	1	2	3
5	$\boxed{4}$	$\boxed{6}$	2	$\boxed{3}$	1
6	5	$\boxed{4}$	3	1	$\boxed{2}$

9.1.1.

$\boxed{1}$	2	3	4	5	6
$\boxed{2}$	1	$\boxed{6}$	5	$\boxed{4}$	3
3	5	$\boxed{1}$	2	$\boxed{6}$	4
4	6	2	1	$\boxed{3}$	$\boxed{5}$
5	$\boxed{3}$	4	6	2	$\boxed{1}$
$\boxed{6}$	4	5	3	1	$\boxed{2}$

10.1.1.

$\boxed{1}$	2	3	$\boxed{4}$	5	6
2	1	$\boxed{6}$	5	$\boxed{4}$	3
$\boxed{3}$	4	$\boxed{1}$	6	2	$\boxed{5}$
4	3	5	$\boxed{1}$	$\boxed{6}$	2
$\boxed{5}$	6	4	2	$\boxed{3}$	$\boxed{1}$
6	5	$\boxed{2}$	3	1	4

10.2.1.

$\boxed{1}$	2	$\boxed{3}$	4	5	6
2	1	4	5	$\boxed{6}$	3
3	4	$\boxed{2}$	6	1	$\boxed{5}$
$\boxed{4}$	5	6	$\boxed{2}$	$\boxed{3}$	1
5	$\boxed{6}$	1	3	$\boxed{2}$	$\boxed{4}$
$\boxed{6}$	3	5	1	$\boxed{4}$	2

11.1.1.

$\boxed{1}$	2	3	4	5	6
2	1	$\boxed{6}$	3	$\boxed{4}$	$\boxed{5}$
$\boxed{3}$	5	4	6	$\boxed{1}$	2
$\boxed{4}$	6	$\boxed{2}$	$\boxed{5}$	3	$\boxed{1}$
5	$\boxed{3}$	1	$\boxed{2}$	6	4
6	4	$\boxed{5}$	1	2	3

11.2.1.

$\boxed{1}$	2	3	4	$\boxed{5}$	6
2	1	$\boxed{5}$	$\boxed{6}$	4	$\boxed{3}$
$\boxed{3}$	5	$\boxed{4}$	2	6	$\boxed{1}$
$\boxed{4}$	6	2	$\boxed{3}$	$\boxed{1}$	5
5	4	6	1	3	$\boxed{2}$
6	3	1	$\boxed{5}$	2	4

12.1.1.

$\boxed{1}$	2	3	4	5	6
2	1	$\boxed{6}$	$\boxed{5}$	3	4
$\boxed{3}$	5	2	$\boxed{6}$	$\boxed{4}$	$\boxed{1}$
$\boxed{4}$	6	$\boxed{5}$	2	1	$\boxed{3}$
5	3	4	1	$\boxed{6}$	2
6	4	1	$\boxed{3}$	$\boxed{2}$	5

12.2.1.

REFERENCES

[1] J. Dénes, A.D. Keedwel, *Latin Squares and their Applications*, Akadémiai Kiadó, Budapest, 1974.