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STRUCTURE CONNECTION IN AN ALMOST CONTACT METRIC MANIFOLD

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Summary. In 1970, semisymetric connection were studied by Yano [1] in a Riemannian manifold and in 1972 Mishra [2] studied affine connection in an almost contact Riemannian manifold. In the present paper we have defined a structure connnection in a Riemannian manifold and studied its properties in an almost contact metric manifold. It is seen that structure connection play fundamental role in an almost contact metric manifold.

1. Introduction

Let M_n be a $n(=2m+1)$ dimensional C^∞-manifold and let there exist a vector valued function f, a vector field t and a l-form A in M_n such that

$$\overline{\overline{X}} + X = A(X)t, \quad \overline{X} \overset{\text{def}}{=} f(X) \tag{1.1}$$

for arbitrary vector field X, then M_n is called an almost contact manifold and the structure (f,t,A) is an almost contact structure. In an almost manifold the following hold [2]

$$\begin{gathered} \operatorname{rank}(f) = n-1, \quad \bar{t} = 0, \quad A(\overline{X}) = 0 \\ A(t) = 1. \end{gathered} \tag{1.2}$$

Let the almost contact manifold M_n be endowed with the nonsingular metric tensor g satisfying

$$g(\overline{X}, \overline{Y}) = g(X,Y) - A(X)A(Y) \tag{1.3}$$

Then M_n is called an a almost contact metric manifold or Grayan manifold.

From (1.3) we obtain

$$g(Y,t) = A(Y) \tag{1.4}$$

Putting $F(X,Y) = g(\overline{X}, Y)$, we have

$$F(\overline{X}, \overline{Y}) = F(X,Y); \quad F(X,Y) = -F(Y,X). \tag{1.5}$$

If D be the Riemannian connection in an almost contact metric manifold, then

$$(D_x A)(Y) = g(D_x t, Y). \tag{1.6}$$

In an almost contact metric manifold Nijenhuis tensor N is given by

$$\begin{aligned} N(X,Y) = N_D(X,Y) &= (D_{\overline{X}} f)(Y) - (D_{\overline{Y}} f)(X) - \\ &- \overline{(D_X f)(Y)} + \overline{(D_Y f)(X)} \end{aligned} \tag{1.7a}$$

$$\begin{aligned} N(X,Y,Z) = N_D(X,Y,Z) &= (D_{\overline{X}} F)(Y,Z) - (D_{\overline{Y}} F)(X,Z) \\ &+ (D_X F)(Y, \overline{Z}) - (D_Y F)(X, \overline{Z}) \end{aligned} \tag{1.7b}$$

where

$$'N(X,Y,Z) = g((D_X F)(Y), Z).$$

An almost contact manifold M_n is said be normal if the almost complex structure J on $M_n \times R$ given by

$$J\left(X, h\frac{d}{dt}\right) = \left(f(X) - ht, A(X)\frac{d}{dt}\right) \tag{1.8}$$

where h is C^∞ – real valued function on M_n, is integrable. From this we have an almost contact manifold is normal if

$$N(X,Y) + dA(X,Y)t = 0. \tag{1.9}$$

An almost contact metric manifold M_n in which

$$F(X,Y) = (D_X A)(Y) - (D_Y A)(X) = (dA)(X,Y) \tag{1.10}$$

is called an almost Sasakian manifold (1.10) is equivalent to

$$(D_X F)(Y,Z) + (D_Y F)(Z,X) + (D_Z F)(X,Y) = 0. \tag{1.11}$$

An almost Sasakian manifold is said to be k-contact Riemannian manifold if A is a Killing vector i.e. if

$$(D_X A)(Y) + (D_Y A)(X) = 0. \tag{1.12}$$

Thus in a K-contact Riemannian manifold we have

$$F(X,Y) = 2(D_X A)(Y) = -2(D_Y A)(X). \tag{1.13}$$

An almost contact manifold with symmetric affine connection D is said to be affinely Sasakian if it is normal and

$$fX = D_X t. \tag{1.14}$$

An almost contast manifold with a symmetric affine connection D is called an affinely almost cosymplectic manifold if

$$D_X f = 0, \qquad D_X A = 0. \tag{1.15}$$

2. Structure Conection in a Riemannian Manifold

Let M_n be a C^∞ – Riemannian manifold of dimension n. Let f^*, t^* and A^* be respectively (1.1) tensor field, a fector field and a l-form in M_n and let D be a Riemannian connection in M_n. We define a connection B in M_n by

$$B_X(Y) = D_X Y + A^*(Y)X - g(X,Y)G_* A^* + F^*(X,Y)t^* \tag{2.1a}$$

and

$$B_X g = 2A^*(X)g, \tag{2.1b}$$
$$g(f^*X, Y) + g(X, f^*Y) = 0 \tag{2.1c}$$

where g is a Riemannian metric and $F^*(X,Y) = g(f^*(X), Y)$ and

$$g(G_* A^*, X) = A^*(X).$$

Definition (2.1). A connection B in a Riemannian manifold M_n given by (2.1a), (2.1b) and (2.1c) is called a structure connection in M_n.

The torsion tensor S of structure connection B is given by

$$S(X,Y) = A^*(Y)X - A^*(X)Y + 2F^*(X,Y)t^*. \tag{2.2}$$

Let us put

$$B_X Y = D_X Y + P(X,Y) \tag{2.3}$$

(2.3) gives

$$S(X,Y) = P(X,Y) - P(Y,X). \tag{2.4}$$

The property (2.1a) of the definition can be written in the form

$$g(P(X,Y),Z) + g(P(X,Z),Y) = -2A^*(X)g(Y,Z). \tag{2.5}$$

Let

$$Q: T(M_n)^* \times T(M_n) \times T(M_n) \to \mathcal{Z}(M_n)$$

be the mapping defined by

$$Q(\omega, X, Y) = g(S(X, G_*\omega), Y) + g(S(Y, G_*\omega), X), \tag{2.6}$$

where $T(M_n)$ is a set of all vector fields on M_n and $T(M_n)^*$ is a set of all l-forms on M_n and $\mathcal{Z}(M)$ is a set of all C^∞ – functions on M_n.

Taking account of (2.2) we have

$$\begin{gathered} Q(\omega, X, Y) = 2A^*(G_*\omega)g(X, Y) - A^*(X)\omega(Y) \\ -A^*(Y)\omega(X) + 2\omega(f^*(X))g(t^*, Y) + 2\omega(f^*Y)g(t^*, X). \end{gathered} \tag{2.7}$$

According (2.4) and (2.5)

$$\begin{aligned} Q(\omega, X, Y) = 2A^*(G_*\omega)g(X, Y) - 2A^*(X)\omega(Y) \\ - 2A^*(Y)\omega(X) - \omega(P(X, Y) + P(Y, X)). \end{aligned} \tag{2.8}$$

From (2.7) and (2.8) we deduce for every l-form

$$\begin{gathered} \omega(P(X, Y) + P(Y, X)) + A^*(X)\omega(Y) + A^*(Y)\omega(X) \\ +2\omega(f^*X)g(t^*, Y) + 2\omega(f^*Y)g(t^*, X) = 0 \end{gathered}$$

and therefore

$$\begin{aligned} P(X, Y) + P(Y, X) + A^*(X)Y + A^*(Y)X + 2f^*(X)g(t^*, Y) \\ + 2f^*(Y)g(t^*, X) = 0 \end{aligned} \tag{2.9}$$

from (2.2) and (2.4) we obtain

$$P(X, Y) - P(Y, X) = A^*(Y)X - A^*(X)Y + 2F^*(X, Y)t^* \tag{2.10}$$

from (2.9) and (2.10) we have

$$\begin{aligned} P(X, Y) = -A^*(X)Y - f^*(X)g(t^*, Y) - f^*(Y)g(t^*, X) \\ + F(X, Y)t^*. \end{aligned} \tag{2.11}$$

Thus from (2.3) and (2.11) structure connection B in a Riemannian manifold (M_n, g) is given by

$$\begin{aligned} B_X(Y) = D_X(Y) - A^*(X)Y - f^*(X)g(t^*, Y) - f^*(Y)g(t^*, X) \\ + F(X, Y)t^*. \end{aligned} \tag{2.12}$$

3. Structure connection in an almost contact meiric manifold

Let M_n be a almost contact metric manifold and (f, t, A) be an almost contact structure on M_n then a contact connection B in an almost contact metric manifold is given by

$$B_X Y = D_X Y - A(X)Y - \overline{X}A(Y) - \overline{Y}A(X) + g(\overline{X}, Y)t \tag{3.1}$$

the structure connection may also be written as

$$(B_X A)Y = (D_X A)Y + A(X)A(Y) - F(X, Y). \tag{3.2}$$

Thus we have the following theorem:

THEOREM 3.1. *Every almost contact metric manifold admits a structure connection B defined by* (3.2). *The structure connection B is uniquely determined by the contact form A and tensor field f.*

From (2.2) we have in an almost contact manifold the following results.

(3.3)
a) $S(\overline{X}, \overline{Y}) = 2F(X, Y)t$
b) $S(X, t) = -\overline{\overline{X}}$
c) $S(\overline{X}, t) = \overline{X}$
d) $S(\overline{\overline{X}}, t) + S(X, t) = 0$
e) $S(\overline{\overline{X}}, Y) = A(X)A(Y)t - A(Y)X - 2F(X, Y)t$
f) $S(\overline{X}, Y) + S(\overline{X}, Y) = A(\overline{Y})X - A(X)Y$
g) $A(S(X, Y)) = 2F(X, Y)$
h) $\overline{\overline{S(X, Y)}} + S(X, Y) - S(\overline{X}, \overline{Y})$

Let us define

$$'S(X, Y, Z) = g(S(X, Y), Z)$$
$$'P(X, Y, Z) = g(P(X, Y), Z).$$

In an almost contact metric manifold with structure connection B we have

$$B_X \overline{Y} = D_X \overline{Y} + g(X, Y)t \tag{3.4a}$$
$$\overline{B}_X \overline{Y} = \overline{D}_X \overline{Y} \tag{3.4b}$$
$$(B_X A)\overline{Y} = (D_X A)\overline{Y} - g(\overline{X}, \overline{Y}) \tag{3.4c}$$
$$(B_X F)(Y, Z) = 2A(X)g(\overline{Y}, Z) + g((B_X f)(Y), Z) \tag{3.4d}$$
$$(B_X f)(Y) = (D_X f)(Y) - g(X, Y)t - A(Y)X \tag{3.4e}$$

THEOREM (3.2). *In an almost contact metric manifold with structure connection B, we have*

$$\begin{aligned} 'P(X,Y,Z) = {} & A(Z)F(X, Y) - A(X)F(Y, Z) \\ & - A(Y)F(X, Z) - A(X)g(Y, Z) \end{aligned} \tag{3.5a}$$
$$'P(\overline{X}, \overline{Y}, \overline{Z}) = 0 =' S(\overline{X}, \overline{Y}, \overline{Z}). \tag{3.5b}$$

PROOF. The proof follows from (1.4) and (2.11).

THEOREM 3.3. *If B a structure connection in an almost contact metric manifold M_n with a Riemannian connection D then*

$$(B_X F)(Y,Z) = (D_X F)(Y,Z) - 2A(X)F(Y,Z) + A(Y)g(X,Z) - A(Z)g(X,Y) \tag{3.6a}$$

$$(B_{\overline{X}} F)(\overline{Y},\overline{Z}) = (D_{\overline{X}} F)(\overline{Y},\overline{Z}). \tag{3.6b}$$

PROOF. We know,

$$X(F(Y,Z)) = (B_X F)(Y,Z) + F(B_X Y, Z) + F(Y, B_X Z)$$

and

$$X(F(Y,Z)) = (D_X F)(Y,Z) + F(D_X Y, Z) + F(Y, D_X Z).$$

From the above equations, we get

$$(B_X F)(Y,Z) = (D_X F)(Y,Z) - {}'P(X,Y,\overline{Z}) + {}'P(X,Z,\overline{Y}). \tag{3.7}$$

Using (3.7), (1.4) and (2.11) we have the theorem.

COROLLARY 1. *If M_n is an almost Sasakian then*

$$(B_X F)(Y,Z) + (B_Y F)(Z,X) + (B_Z F)(X,Y) = -2A(X)F(Y,Z) - 2A(Y)F(Z,X) - 2A(Z)F(X,Y). \tag{3.8}$$

COROLLARY 2. *In an almost contact metric manifold with structure connection B we have*

$$(B_{\overline{X}} F)(\overline{Y},Z) = (D_{\overline{X}} F)(\overline{Y},Z) - A(Z)g(X,Y) + A(X)A(Y)A(Z) \tag{3.9a}$$

$$(B_X F)(\overline{\overline{Y}},\overline{\overline{Z}}) + (B_X F)(\overline{Y},\overline{Z}) = -4A(X)F(Y,Z) \tag{3.9b}$$

THEOREM 3.4. *In an almost contact metric manifold with a structure connection B we have*

$$N_B(X,Y) = N(X,Y) + S(\overline{X},\overline{Y}) \tag{3.10a}$$

$${}'N(X,Y,Z) = {}'N_B(X,Y,Z) + 2A(Y)g(X,Z) - 2A(X)g(Y,Z) \tag{3.10b}$$

where

$$N_B(X,Y) = (B_{\overline{X}} f)(Y) - (B_{\overline{Y}} f)(X) - (\overline{B_X f})(Y) + (\overline{B_Y f})(X).$$

PROOF. We have in an almost contact metric manifold Nijenhuis tensor N is given by

$$\begin{aligned}
N(X,Y) &= D_{\overline{X}}\overline{Y} - \overline{D_{\overline{X}}Y} - D_{\overline{Y}}\overline{X} + \overline{D_{\overline{Y}}X} - \overline{D_X\overline{\overline{Y}}} + \overline{\overline{D_X Y}} + \overline{D}_Y\overline{X} - \overline{\overline{D}}_Y X \\
&= B_{\overline{X}}\overline{Y} - F(X,Y)t - B_{\overline{Y}}\overline{X} + F(Y,X)t - \overline{B}_{\overline{X}}Y \\
&= \overline{\overline{\overline{X}}}A(Y) + \overline{B}_{\overline{Y}}X + \overline{\overline{\overline{Y}}}A(X) - \overline{B}_{\overline{X}}\overline{Y} - A(X)\overline{\overline{Y}} \\
&\quad - \overline{\overline{\overline{Y}}}A(X) + \overline{B}_Y\overline{X} + A(Y)\overline{\overline{X}} + \overline{\overline{\overline{X}}}A(Y) + \overline{\overline{B}}_X Y \\
&\quad + A(X)\overline{\overline{Y}} + \overline{\overline{\overline{X}}}A(Y) + \overline{\overline{\overline{Y}}}A(X) - \overline{\overline{B}}_Y X - A(Y)\overline{\overline{X}} \\
&\quad - \overline{\overline{\overline{Y}}}A(X) - \overline{\overline{\overline{X}}}A(Y).
\end{aligned}$$

From this have the theorem (3.4).

THEOREM 3.5. *In a k-contact Riemannian manifold with structure connection B we have*

$$(B_X A)(Y) + (B_Y A)(X) = 2A(X)A(Y) \tag{3.11a}$$

$$A(N_B(X,Y)) = F(X,Y). \tag{3.11b}$$

PROOF. Using (1.12) and (3.2) we get first part of the theorem.

For the proof of $2nd$, using the equations (1.13) and (3.2) we have in a k-contact Riemannian manifold.

$$(B_{\overline{X}}A)(\overline{Y}) - (B_{\overline{Y}}A)(\overline{X}) = -F(X,Y) \tag{3.12}$$

also

$$(B_{\overline{X}}A)(\overline{Y}) = -A((B_{\overline{X}}f)(Y)). \tag{3.13}$$

Again from (1.17) a, we have

$$A(N_B(\overline{X},\overline{Y})) = A((B_{\overline{X}}f)(Y)) - A((B_{\overline{Y}}f)(X)) \tag{3.14}$$

from (3.12), (3.13) and (3.14) we have the $2nd$ part of the theorem.

COROLLARY. *In a k-contact Riemannian manifold with structure connection B, we have*

$$A(N_B(\overline{X},\overline{Y})) = A(N_B(X,Y)) \tag{3.15a}$$

$$A(N_B(\overline{\overline{X}},\overline{Y})) + A(N_B(\overline{X},\overline{\overline{Y}})) = 0. \tag{3.15b}$$

Theorem 3.6 *In a normal contact metric manifold with structure connection B*

$$N_B(X,Y) = F(X,Y)t = \frac{1}{2}S(\overline{X},\overline{Y}). \tag{3.6}$$

Proof. Using (1.9) (3.3a) and (3.10a), we have the theorem.

Theorem 3.7. *In a k-contact RIemannian manifold with structure connection B, we have*

$$B_X t = -\,A(X)t \tag{3.17a}$$

$$(B_X A)(Y) = A(X)A(Y). \tag{3.17b}$$

Proof. In a k-contact manifold we can easily obtain

$$D_X t = \overline{X} \tag{3.18}$$

(3.18) together with (3.1) gives the first part of the theorem. Using (3.2) and (1.13) we have $2nd$ part of the theorem.

Corollary. *An almost contact metric manifold M_n with structure connection B is a affinely Sasakian manifold if*

$$A(N_B(X,Y)) = F(X,Y) \tag{3.19a}$$

and

$$B_X t = -A(X)t. \tag{3.19b}$$

4. Curvature of structure connection in an almost contact metric manifold

Let $\tilde{R}$ be the curvature tensor of structure connection B and K be the curvature tensor of Riemannian connection D in an almost metric manifold, then

$$\tilde{R}(X,Y,Z) = B_X B_Y Z - B_Y B_X Z - B_{[X,Y]}Z. \tag{4.1}$$

We have following theorem:

Theorem (4.1) *In an almost contact metric manifold the curvature tensor $\tilde{R}$ is given by*

$$\tilde{R}(X,Y,Z) = K(X,Y,Z) + {}_X\theta_Y - {}_Y\theta_X, \tag{4.2}$$

where ${}_X\theta_Y$ is given by

$$
\begin{aligned}
{}_X\theta_Y &= (D_Y A)(X)(Z+\overline{Z}) + (D_X F)(Y,Z)t \\
&+ F(Y,Z)D_X t + A(Y)A(Z)X + (D_Y A)(Z)\overline{X} \\
&+ F(X,Z)\overline{Y} + A(X)g(Y,Z)t + A(Z)(D_Y f)(X) \\
&+ A(X)(D_Y f)(Z).
\end{aligned} \tag{4.3}
$$

THEOREM (4.2). *In an affinely almost cosympletic manifold we have*

$$
\begin{aligned}
\tilde{R}(X,Y,Z) = K(X,Y,Z) &+ A(Y)A(Z)X - A(X)A(Z)Y \\
&- F(Y,Z)\overline{X} + F(X,Z)\overline{Y} + A(X)g(Y,Z)t \\
&- A(Y)g(X,Z)t
\end{aligned} \tag{4.4a}
$$

$$(C_1^1\tilde{R})(Y,Z) = \mathrm{Ric}\,(Y{,}Z) + (n-1)A(Y)A(Z) \tag{4.4b}$$

$$C_1^1 E = r + \mathrm{rank}\,(f) \tag{4.4c}$$

where $r = C_1^1 R$ is scalar tensor and

$$
\begin{aligned}
\mathrm{Ric}\,(Y,Z) &= g(R(Y),Z) \\
(C_1^1\tilde{R}(Y,Z) &= g(E(Y),Z).
\end{aligned} \tag{4.5}
$$

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