STRUCTURE CONNECTION IN AN ALMOST CONTACT METRIC MANIFOLD

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Summary. In 1970, semi-symmetric connection were studied by Yano [1] in a Riemannian manifold and in 1972 Mishra [2] studied affine connection in an almost contact Riemannian manifold. In the present paper we have defined a structure connection in a Riemannian manifold and studied its properties in an almost contact metric manifold. It is seen that structure connection play fundamental role in an almost contact metric manifold.

1. Introduction

Let $M_n$ be a $n (= 2m + 1)$ dimensional $C^\infty$-manifold and let there exist a vector valued function $f$, a vector field $t$ and a $l$-form $A$ in $M_n$ such that

$$\nabla X + X = A(X)t, \quad \nabla \overset{\text{def}}{=} f(X)$$

for arbitrary vector field $X$, then $M_n$ is called an almost contact manifold and the structure $(f, t, A)$ is an almost contact structure. In an almost manifold the following hold [2]

$$\text{rank} (f) = n - 1, \quad \overset{\text{def}}{t} = 0, \quad A(\nabla) = 0$$

$$A(t) = 1.$$

Let the almost contact manifold $M_n$ be endowed with the nonsingular metric tensor $g$ satisfying

$$g(\nabla, \nabla) = g(X, Y) - A(X)A(Y)$$

Then $M_n$ is called an a almost contact metric manifold or Grayan manifold.

From (1.3) we obtain

$$g(Y, t) = A(Y)$$
Putting $F(X, Y) = g(\overline{X}, Y)$, we have

(1.5) \[ F(\overline{X}, \overline{Y}) = F(X, Y); \quad F(X, Y^{'}) = -F(Y, X). \]

If $D$ be the Riemannian connection in an almost contact metric manifold, then

(1.6) \[ (D_X A)(Y) = g(D_X t, Y). \]

In an almost contact metric manifold Nijenhuis tensor $N$ is given by

(1.7a) \[ N(X, Y) = N_D(X, Y) = (D_X f)(Y) - (D_Y f)(X) - \]

\[ - (D_X f)(Y) + (D_Y f)(X) \]

(1.7b) \[ N(X, Y, Z) = N_D(X, Y, Z) = (D_X F)(Y, Z) - (D_Y F)(X, Z) + (D_X F)(Y, \overline{Z}) - (D_Y F)(X, \overline{Z}) \]

where

\[ 'N(X, Y, Z) = g((D_X F)(Y), Z). \]

An almost contact manifold $M_n$ is said be normal if the almost complex structure $J$ on $M_n \times R$ given by

(1.8) \[ J \left( X, h \frac{d}{dt} \right) = \left( f(X) - h t, A(X) \frac{d}{dt} \right) \]

where $h$ is $C^\infty$ - real valued function on $M_n$, is integrable. From this we have an almost contact manifold is normal if

(1.9) \[ N(X, Y) + dA(X, Y)t = 0. \]

An almost contact metric manifold $M_n$ in which

(1.10) \[ F(X, Y) = (D_X A)(Y) - (D_Y A)(X) = (dA)(X, Y) \]

is called an almost Sasakian manifold (1.10) is equivalent to

(1.11) \[ (D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y) = 0. \]

An almost Sasakian manifold is said to be $k$-contact Riemannian manifold if $A$ is a Killing vector i.e. if

(1.12) \[ (D_X A)(Y) + (D_Y A)(X) = 0. \]

Thus in a $K$-contact Riemannian manifold we have

(1.13) \[ F(X, Y) = 2(D_X A)(Y) = -2(D_Y A)(X). \]
An almost contact manifold with symmetric affine connection $D$ is said to be affinely Sasakian if it is normal and

\[(1.14) \quad fX = D_X t.\]

An almost contact manifold with a symmetric affine connection $D$ is called an affinely almost cosymplectic manifold if

\[(1.15) \quad D_X f = 0, \quad D_X A = 0.\]

2. Structure Connection in a Riemannian Manifold

Let $M_n$ be a $C^\infty$ - Riemannian manifold of dimension $n$. Let $f^*$, $t^*$ and $A^*$ be respectively (1.1) tensor field, a vector field and a $1$-form in $M_n$ and let $D$ be a Riemannian connection in $M_n$. We define a connection $B$ in $M_n$ by

\[(2.1a) \quad B_X (Y) = D_X Y + A^*(Y)X - g(X, Y)G_*A^* + F^*(X, Y)t^*\]

and

\[(2.1b) \quad B_X g = 2A^*(X)g,\]
\[(2.1c) \quad g(f^*X, Y) + g(X, f^*Y) = 0\]

where $g$ is a Riemannian metric and $F^*(X, Y) = g(f^*(X), Y)$ and

\[g(G_*A^*, X) = A^*(X).\]

**Definition (2.1).** A connection $B$ in a Riemannian manifold $M_n$ given by (2.1a), (2.1b) and (2.1c) is called a structure connection in $M_n$.

The torsion tensor $S$ of structure connection $B$ is given by

\[(2.2) \quad S(X, Y) = A^*(Y)X - A^*(X)Y + 2F^*(X, Y)t^*.\]

Let us put

\[(2.3) \quad B_X Y = D_X Y + P(X, Y)\]

(2.3) gives

\[(2.4) \quad S(X, Y) = P(X, Y) - P(Y, X).\]

The property (2.1a) of the definition can be written in the form

\[(2.5) \quad g(P(X, Y), Z) + g(P(X, Z), Y) = -2A^*(X)g(Y, Z).\]
Let

\[ Q: T(M_n)^* \times T(M_n) \times T(M_n) \to \mathcal{Z}(M_n) \]

be the mapping defined by

(2.6) \[ Q(\omega, X, Y) = g(S(X, G, \omega), Y) + g(S(Y, G, \omega), X) \]

where \( T(M_n) \) is a set of all vector fields on \( M_n \) and \( T(M_n)^* \) is a set of all \( l \)-forms on \( M_n \), and \( \mathcal{Z}(M) \) is a set of all \( C^\infty \) functions on \( M_n \).

Taking account of (2.2) we have

(2.7) \[
\begin{align*}
Q(\omega, X, Y) &= 2 A^*(G, \omega)g(X, Y) - A^*(X)\omega(Y) \\
&- A^*(Y)\omega(X) + 2\omega(f^*(X))g(t^*, Y) + 2\omega(f^*Y)g(t^*, X).
\end{align*}
\]

According (2.4) and (2.5)

(2.8) \[
\begin{align*}
Q(\omega, X, Y) &= 2 A^*(G, \omega)g(X, Y) - 2 A^*(X)\omega(Y) \\
&- 2 A^*(Y)\omega(X) - \omega(P(X, Y) + P(Y, X)).
\end{align*}
\]

From (2.7) and (2.8) we deduce for every \( l \)-form

\[
\begin{align*}
\omega(P(X, Y) + P(Y, X)) + A^*(X)\omega(Y) + A^*(Y)\omega(X) \\
&+ 2\omega(f^*X)g(t^*, Y) + 2\omega(f^*Y)g(t^*, X) = 0
\end{align*}
\]

and therefore

(2.9) \[
\begin{align*}
P(X, Y) + P(Y, X) + A^*(X)Y + A^*(Y)X + 2f^*(X)g(t^*, Y) \\
+ 2f^*(Y)g(t^*, X) = 0
\end{align*}
\]

from (2.2) and (2.4) we obtain

(2.10) \[
P(X, Y) - P(Y, X) = A^*(Y)X - A^*(X)Y + 2F^*(X,Y)t^*
\]

from (2.9) and (2.10) we have

(2.11) \[
P(X, Y) = - A^*(X)Y - f^*(X)g(t^*, Y) - f^*(Y)g(t^*, X) \\
+ F(X, Y)t^*.
\]

Thus from (2.3) and (2.11) structure connection \( B \) in a Riemannian manifold \( (M_n, g) \) is given by

(2.12) \[
B_X(Y) = D_X(Y) - A^*(X)Y - f^*(X)g(t^*, Y) - f^*(Y)g(t^*, X) \\
+ F(X, Y)t^*.
\]
3. Structure connection in an almost contact metric manifold

Let \( M_n \) be a almost contact metric manifold and \((f, t, A)\) be an almost contact structure on \( M_n \) then a contact connection \( B \) in an almost contact metric manifold is given by

\[
(3.1) \quad B_X Y = D_X Y - A(X)Y - \nabla A(Y) - \nabla A(X) + g(\nabla, Y)t
\]

the structure connection may also be written as

\[
(3.2) \quad (B_X A)Y = (D_X A)Y + A(X)A(Y) - F(X, Y).
\]

Thus we have the following theorem:

**Theorem 3.1.** Every almost contact metric manifold admits a structure connection \( B \) defined by (3.2). The structure connection \( B \) is uniquely determined by the contact form \( A \) and tensor field \( f \).

From (2.2) we have in an almost contact manifold the following results.

\[
(3.3) \quad \begin{align*}
\text{a) } & S(\nabla, \nabla) = 2F(X, Y)t \\
\text{b) } & S(X, t) = -\nabla \\
\text{c) } & S(\nabla, t) = \nabla \\
\text{d) } & S(\nabla, t) + S(X, t) = 0 \\
\text{e) } & S(\nabla, Y) = A(X)A(Y)t - A(Y)X - 2F(X, Y)t \\
\text{f) } & S(\nabla, Y) + S(\nabla, Y) = A(\nabla)X - A(X)Y \\
\text{g) } & A(S(X, Y)) = 2F(X, Y) \\
\text{h) } & S(\nabla, Y) + S(X, Y) = S(\nabla, \nabla)
\end{align*}
\]

Let us define

\[ \iota S(X, Y, Z) = g(S(X, Y), Z) \]

\[ \iota P(X, Y, Z) = g(P(X, Y), Z). \]

In an almost contact metric manifold with structure connection \( B \) we have

\[
(3.4a) \quad B_X \nabla = D_X \nabla + g(X, Y)t \\
(3.4b) \quad B_X \nabla = D_X \nabla \\
(3.4c) \quad (B_X A)\nabla = (D_X A)\nabla - g(\nabla, \nabla) \\
(3.4d) \quad (B_X F)(Y, Z) = 2A(X)g(\nabla, Z) + g((B_X f)(Y), Z) \\
(3.4e) \quad (B_X F)(Y) = (D_X f)(Y) - g(X, Y)t - A(Y)X
\]

**Theorem (3.2).** In an almost contact metric manifold with structure connection \( B \), we have

\[
(3.5a) \quad \iota P(X, Y, Z) = A(Z)F(X, Y) - A(X)F(Y, Z) - A(Y)F(X, Z) - A(X)g(Y, Z) \\
(3.5b) \quad \iota P(\nabla, \nabla, Z) = 0 = \iota S(\nabla, \nabla, Z).
\]
\textbf{Proof.} The proof follows from (1.4) and (2.11).

\textbf{Theorem 3.3.} If $B$ a structure connection in an almost contact metric manifold $M_n$ with a Riemannian connection $D$ then

\begin{align}
(B_X F)(Y, Z) &= (D_X F)(Y, Z) - 2A(X)F(Y, Z) \\
&\quad + A(Y)g(X, Z) - A(Z)g(X, Y)
\end{align}

\begin{align}
(B_X \bar{F})(Y, Z) &= (D_X \bar{F})(Y, Z) - \bar{P}(X, Y, \bar{Z}) + \bar{P}(X, Z, \bar{Y}).
\end{align}

Proof. We know,

\begin{align}
X(F(Y, Z)) &= (B_X F)(Y, Z) + F(B_X Y, Z) + F(Y, B_X Z)
\end{align}

and

\begin{align}
X(F(Y, Z)) &= (D_X F)(Y, Z) + F(D_X Y, Z) + F(Y, D_X Z).
\end{align}

From the above equations, we get

\begin{align}
(B_X F)(Y, Z) &= (D_X F)(Y, Z) - \bar{P}(X, Y, \bar{Z}) + \bar{P}(X, Z, \bar{Y}).
\end{align}

Using (3.7), (1.4) and (2.11) we have the theorem.

\textbf{Corollary 1.} If $M_n$ is an almost Sasakian then

\begin{align}
(B_X F)(Y, Z) + (B_\bar{Y} F)(Z, X) + (B_\bar{Z} F)(X, Y) &= -2A(X)F(Y, Z) \\
&\quad - 2A(Y)F(Z, X) - 2A(Z)F(X, Y).
\end{align}

\textbf{Corollary 2.} In an almost contact metric manifold with structure connection $B$ we have

\begin{align}
(B_X \bar{F})(Y, Z) &= (D_X \bar{F})(Y, Z) \\
&\quad - A(Z)g(X, Y) + A(X)A(Y)A(Z)
\end{align}

\begin{align}
(B_X F)(Y, Z) + (B_X \bar{F})(Y, \bar{Z}) &= -4A(X)F(Y, Z)
\end{align}

\textbf{Theorem 3.4.} In an almost contact metric manifold with a structure connection $B$ we have

\begin{align}
N_B(X, Y) &= N(X, Y) + S(\bar{X}, \bar{Y})
\end{align}

\begin{align}
\bar{N}(X, Y, Z) &= \bar{N}_B(X, Y, Z) + 2A(Y)g(X, Z) - 2A(X)g(Y, Z)
\end{align}

where

\begin{align}
N_B(X, Y) &= (B_X f)(Y) - (B_X \bar{f})(X) - (B_X \bar{f})(Y) + (B_Y \bar{f})(X).
\end{align}
Proof. We have in an almost contact metric manifold Nijenhuis tensor $N$ is given by

$$N(X,Y) = D_XY - D_XF - D_YX + D_YF - D_XY + D_YX - D_YX$$

$$= B_XY - F(X,Y)t - B_YX + F(Y,X)t - D_XY$$

$$= XA(Y) + B_YX + YA(X) - B_YX - A(X)Y$$

$$- YA(X) + B_YX + A(Y)X + XA(Y) + B_XY$$

$$+ A(X)Y + XA(Y) + YA(X) - B_YX - A(Y)X$$

$$- YA(X) - XA(Y).$$

From this have the theorem (3.4).

Theorem 3.5. In a k-contact Riemannian manifold with structure connection $B$ we have

$$\begin{align*}
(3.11a) & \quad (B_XA)(Y) + (B_YA)(X) = 2A(X)A(Y) \\
(3.11b) & \quad A(N_B(X,Y)) = F(X,Y).
\end{align*}$$

Proof. Using (1.12) and (3.2) we get first part of the theorem.

For the proof of 2nd, using the equations (1.13) and (3.2) we have in a k-contact Riemannian manifold.

$$\begin{align*}
(3.12) & \quad (B_XA)(Y) - (B_YA)(X) = -F(X,Y)
\end{align*}$$

also

$$\begin{align*}
(3.13) & \quad (B_XA)(Y) = -A((B_Xf)(Y)).
\end{align*}$$

Again from (1.17) a, we have

$$\begin{align*}
(3.14) & \quad A(N_B(X,Y)) = A((B_Xf)(Y)) - A((B_Yf)(X))
\end{align*}$$

from (3.12), (3.13) and (3.14) we have the 2nd part of the theorem.

Corollary. In a k-contact Riemannian manifold with structure connection $B$, we have

$$\begin{align*}
(3.15a) & \quad A(N_B(X,Y)) = A(N_B(X,Y)) \\
(3.15b) & \quad A(N_B(X,Y)) + A(N_B(X,Y)) = 0.
\end{align*}$$
Theorem 3.6. In a normal contact metric manifold with structure connection $B$

\[(3.6) \quad N_B(X,Y) = F(X,Y)t = \frac{1}{2}S(\overline{X},\overline{Y}).\]

Proof. Using (1.9) (3.3a) and (3.10a), we have the theorem.

Theorem 3.7. In a $k$-contact Riemannian manifold with structure connection $B$, we have

\[(3.17a) \quad B_X t = -A(X)t \]
\[(3.17b) \quad (B_X A)(Y) = A(X)A(Y).\]

Proof. In a $k$-contact manifold we can easily obtain

\[(3.18) \quad D_X t = \overline{X}\]

(3.18) together with (3.1) gives the first part of the theorem. Using (3.2) and (1.13) we have 2nd part of the theorem.

Corollary. An almost contact metric manifold $M_n$ with structure connection $B$ is a affinely Sasakian manifold if

\[(3.19a) \quad A(N_B(X,Y)) = F(X,Y)\]

and

\[(3.19b) \quad B_X t = -A(X)t.\]

4. Curvature of structure connection in an almost contact metric manifold

Let $\tilde{R}$ be the curvature tensor of structure connection $B$ and $K$ be the curvature tensor of Riemannian connection $D$ in an almost metric manifold, then

\[(4.1) \quad \tilde{R}(X,Y,Z) = B_X B_Y Z - B_Y B_X Z - B_{[X,Y]} Z.\]

We have following theorem:

Theorem (4.1) In an almost contact metric manifold the curvature tensor $\tilde{R}$ is given by

\[(4.2) \quad \tilde{R}(X,Y,Z) = K(X,Y,Z) + X\theta_Y - Y\theta_X,\]
where $\chi\theta_Y$ is given by

$$
\chi\theta_Y = (D_Y A)(X)(Z + \bar{Z}) + (D_X F)(Y, Z)t \\
+ F(Y, Z)D_X t + A(Y)A(Z)X + (D_Y A)(Z)\bar{X} \\
+ F(X, Z)\bar{Y} + A(X)g(Y, Z)t + A(Z)(D_Y f)(X) \\
+ A(X)(D_Y f)(Z).
$$

**Theorem (4.2).** In an affinely almost cosymplectic manifold we have

$$
(4.4a) \quad \tilde{R}(X, Y, Z) = K(X, Y, Z) + A(Y)A(Z)X - A(X)A(Z)Y \\
- F(Y, Z)\bar{X} + F(X, Z)\bar{Y} + A(X)g(Y, Z)t \\
- A(Y)g(X, Z)t
$$

$$
(4.4b) \quad (C_1^1 \tilde{R})(Y, Z) = \text{Ric}(Y, Z) + (n - 1)A(Y)A(Z)
$$

$$
(4.4c) \quad C_1^1 E = r + \text{rank}(f)
$$

where $r = C_1^1 R$ is scalar tensor and

$$
(4.5) \quad \text{Ric}(Y, Z) = g(R(Y), Z)
$$

$$
(4.5) \quad (C_1^1 \tilde{R})(Y, Z) = g(E(Y), Z).
$$

**REFERENCES**


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