

## ON INTEGRABILITY OF TRIGONOMETRIC SERIES WITH QUASI-MONOTONE COEFFICIENTS

*Tatjana Ostrogorski*

### 1. Introduction and results

Consider the formal sine and cosine series

$$g(x) = \sum_1^{\infty} a_n \sin nx \quad \text{and} \quad f(x) = \frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nx.$$

The following problem has been studied by many authors: if  $\psi$  is a given positive function, what hypothesis on  $\{a_n\}$  are equivalent to  $g\psi \in L(0, \pi)$  or to  $f\psi \in L(0, \pi)$ ?

First assume that the sequence  $\{a_n\}$  is monotonely decreasing to zero as  $n \rightarrow \infty$ .

Boas [4] proved that for  $\psi(x) = x^{-\gamma}$ ,  $0 < \gamma < 2$ , the following holds

$$(1) \quad g\psi \in L(0, \pi) \Leftrightarrow \sum_1^{\infty} \frac{\psi(n)}{n} a_n < \infty.$$

Aljančić, Bojanić, Tomić [1] proved that assertion (1) holds true also for  $\psi(x) = x^{-\gamma}L\left(\frac{1}{x}\right)$ ,  $0 < \gamma < 2$ , where  $L$  is a slowly varying function; and Aljančić [3] obtained a similiar result for  $\psi(x) = K\left(\frac{1}{x}\right)$  where  $K \in \mathcal{K}$  ( $0 < \underline{\rho}, \bar{\rho} < 2$ ).

Here  $\mathcal{K}(\underline{\rho}, \bar{\rho})$  denotes the class of function defined on  $I_a = [a, \infty)$ ,  $a > 0$ , which are *0-regularly varying* (0-RV) at infinity with lower index  $\underline{\rho}$  and upper index  $\bar{\rho}$  [2].

More generally, suppose that the sequence  $\{a_n\}$  is *quasi-monotone*, i.e.  $a_n \geq 0$  and for some  $\alpha > 0$

$$a_{n+1} \leq a_n \left(1 + \frac{\alpha}{n}\right),$$

for  $n$  large enough [8].

Shah [6] and Yong [7] proved, for  $\psi(x) = x^{-\gamma}$  and  $\psi(x) = x^{-\gamma}L\left(\frac{1}{x}\right)$ ,  $0 < \gamma < 2$ , respectively, that formula (1) still remains true if  $\{a_n\}$  is quasi-monotone and tending to zero as  $n \rightarrow \infty$ .

Moreover, Igari [5] and Yong [7] considered the integrability of  $\psi g^p$ , where  $p \geq 1$ .

Similar results are valid for  $f$ .

In the present paper we combine the before mentioned results to prove (1) for both  $\{a_n\}$  quasi-monotone and  $\psi$  0-regularly varying.

**THEOREM 1.** *Let  $\{a_n\}$  be quasi-monotone and  $a_n \rightarrow 0$ ,  $n \rightarrow \infty$ , and let  $K \in \mathcal{K}(1 - p < \underline{\rho}, \bar{\rho} < 1 + p)$ , for  $1 \leq p < \infty$ . Then*

$$K\left(\frac{1}{x}\right)g^p(x) \in L(0, \pi) \Leftrightarrow \sum_1^{\infty} n^{p-2}K(n)a_n^p < \infty.$$

**THEOREM 2.** *Let  $\{a_n\}$  be quasi-monotone and  $a_n \rightarrow 0$ ,  $n \rightarrow \infty$ , and let  $K \in \mathcal{K}(1 - p < \underline{\rho}, \bar{\rho} < 1)$ , for  $1 \leq p < \infty$ . Then*

$$K\left(\frac{1}{x}\right)f^p(x) \in L(0, \pi) \Leftrightarrow \sum_1^{\infty} n^{p-2}K(n)a_n^p < \infty.$$

**REMARK.** These theorems give on one hand, a generalisation of Theorems 2 and 3 [3], in which the sequence  $\{a_n\}$  is monotone and  $p = 1$ , and on the other hand, a generalisation of Theorems 1 and 2 [7], in which the function  $K$  is regularly varying and the sequence  $\{a_n\}$  satisfies the further assumption

$$(*) \quad 0 < M_1 \leq n^\beta L(n)a_n \leq M_2 < \infty, \quad \beta > 0,$$

where  $L$  is a slowly varying function and  $M_1, M_2$  are constants.

In the proof of the theorems we shall have need of the following lemmas. They are generalisations of Theorem 1 [5] to 0-regularly varying functions.

By  $C$ , possibly with subscripts, we denote a constant; a  $C$  may stay for different constants from one appearance to another.

**LEMMA 1.** *Let  $K \in \mathcal{K}(\underline{\rho}, \bar{\rho} < p - 1)$ , for  $1 \leq p < \infty$ .*

1) *Let  $f \geq 0$  be locally integrable on  $I_a$  and  $F(t) = \int_a^t f(u)du$ . Then*

$$\int_a^{\infty} K(t) \left(\frac{F(t)}{t}\right)^p dt \leq C \int_a^{\infty} K(t)f^p(t)dt.$$

2) Let  $c_n \geq 0$  and  $A_n = \sum_{k=1}^n c_k$ . Then

$$\sum_1^\infty K(n) \left( \frac{A_n}{n} \right)^p \leq C \sum_1^\infty K(n) c_n^p.$$

LEMMA 2. Let  $K \in \mathcal{K}(p-1 < \underline{\rho}, \bar{\rho})$ , for  $1 \leq p < \infty$ .

1) Let  $f \geq 0$  be locally integrable on  $I_a$  and  $G(t) = \int_t^\infty f(u)du$ . Then

$$\int_a^\infty K(t) \left( \frac{G(t)}{t} \right)^p dt \leq C \int_a^\infty K(t) f^p(t) dt.$$

2) Let  $c_n \geq 0$  and  $B_n = \sum_{k=n}^\infty c_k$ . Then

$$\sum_1^\infty K(n) \left( \frac{B_n}{n} \right)^p \leq C \sum_1^\infty K(n) c_n^p.$$

COROLLARY 1. Let  $K \in \mathcal{K}(\underline{\rho}, \bar{\rho} < 1-p)$ , for  $1 \leq p < \infty$ . Let  $f \geq 0$  be locally integrable in  $(0, 1/a)$  and  $G(t) = \int_t^{1/a} f(u)du$ . Then

$$\int_0^{1/a} K \left( \frac{1}{t} \right) \left( \frac{G(t)}{t} \right)^p dt \leq C \int_0^{1/a} K \left( \frac{1}{t} \right) f^p(t) dt.$$

COROLLARY 2. Let  $K \in \mathcal{K}(1-p < \underline{\rho}, \bar{\rho})$ , for  $1 \leq p < \infty$ . Let  $f \geq 0$  be locally integrable in  $(0, 1/a)$  and  $F(t) = \int_0^t f(u)du$ . Then

$$\int_0^{1/a} K \left( \frac{1}{t} \right) \left( \frac{F(t)}{t} \right)^p dt \leq C \int_0^{1/a} K \left( \frac{1}{t} \right) f^p(t) dt.$$

## 2. Properties of 0-regularly varying functions

In this section we cite some properties of 0-regularly varying functions that are needed in the proof of the theorems. As before,  $K$  is assumed to belong to the class  $\mathcal{K}(\underline{\rho}, \bar{\rho})$ .

(i-1) if  $\sigma < \underline{\rho}$  then  $\int_a^x t^{-\sigma-1} K(t) dt \leq C_1 x^{-\sigma} K(x)$ .

2) if  $\tau > \bar{\rho}$  then  $\int_x^\infty t^{-\tau-1} K(t) dt \leq C_2 x^{-\tau} K(x)$ .

(ii) There exist positive constants  $C_1, C_2$  such that

$$C_1 n^{-2} K(n) \leq \int_{1/n}^{1/(n-1)} K\left(\frac{1}{x}\right) dx \leq C_2 n^{-2} K(n).$$

(iii) If  $\sigma < \rho$  then for  $g$  locally integrable in  $(0, \infty)$

$$1) \int_a^\infty x^\sigma g(x) dx \leq C_1 \int_a^\infty g(x) K(x) dx$$

$$2) \int_0^{1/a} x^{-\sigma} g(x) dx \leq C_2 \int_0^{1/a} g(x) K\left(\frac{1}{x}\right) dx.$$

### 3. Proofs of the theorems

PROOF OF THEOREM 1. To prove part  $\Leftarrow$  consider

$$|g(x)| = \left| \sum_{k=1}^{\infty} a_k \sin kx \right| \leq \sum_{k=1}^n a_k kx + \left| \sum_{k=n+1}^{\infty} a_k \sin kx \right|.$$

An application of Abel's transformation to the second sum yields

$$\begin{aligned} |g(x)| &\leq x \sum_{k=1}^n k a_k + \left| \frac{1}{2 \sin \frac{x}{2}} \sum_{k=n}^{\infty} (a_k - a_{k+1}) \left( \cos \left( n + \frac{1}{2} \right) x - \cos \left( k + \frac{1}{2} \right) x \right) \right| \leq \\ &\leq x \sum_{k=1}^n k a_k + \frac{\pi}{x} \sum_{k=n}^{\infty} |a_k - a_{k+1}|. \end{aligned}$$

If we put  $\frac{\pi}{n} < x \leq \frac{\pi}{n-1}$  then we have

$$(2) \quad |g(x)| \leq \frac{\pi}{n-1} \sum_{k=1}^n k a_k + n \sum_{k=n}^{\infty} |a_k - a_{k+1}|.$$

The sequence  $\{a_n\}$  being quasi-monotone, it follows by [8, p. 5]

$$(3) \quad \sum_{k=n}^{\infty} |a_k - a_{k+1}| \leq a_n + 2\alpha \sum_{k=n}^{\infty} \frac{a_k}{k}.$$

Therefore we obtain by substituting (3) into (2)

$$|g(x)| \leq \frac{\pi}{n-1} \sum_{k=1}^n k a_k + n \left( a_n + 2\alpha \sum_{k=n}^{\infty} \frac{a_k}{k} \right), \quad \frac{\pi}{n} < x \leq \frac{\pi}{n-1},$$

and introducing the notation  $A_n = \sum_{k=1}^n ka_k$ ,  $B_n = \sum_{k=n}^{\infty} \frac{a_k}{k}$ , we have

$$(4) \quad |g(x)| \leq C \left( \frac{1}{n} A_n + na_n + nB_n \right), \quad \frac{\pi}{n} < x \leq \frac{\pi}{n-1}.$$

Now, this implies that

$$(5) \quad \begin{aligned} \int_0^{\pi} |g(x)|^p K\left(\frac{1}{x}\right) dx &= \sum_{n=1}^{\infty} \int_{\pi/n}^{\pi/(n-1)} |g(x)|^p K\left(\frac{1}{x}\right) dx \leq \\ &\leq C \sum_{n=1}^{\infty} \left( \frac{1}{n} A_n + na_n + nB_n \right)^p \int_{\pi/n}^{\pi/(n-1)} K\left(\frac{1}{x}\right) dx \leq \\ &\leq C_1 \sum_{n=1}^{\infty} \left( \frac{1}{n} A_n + na_n + nB_n \right)^p n^{-2} K(n) \end{aligned}$$

(where for the last inequality we have used property (ii) of 0-RV functions).

An application of Minkowsky's inequality yields

$$(6) \quad \begin{aligned} \left[ \int_0^{\pi} |g(x)|^p K\left(\frac{1}{x}\right) dx \right]^{1/p} &\leq C_1 \left[ \sum_{n=1}^{\infty} \left( \frac{1}{n} A_n + na_n + nB_n \right)^p n^{-2} K(n) \right]^{1/p} \leq \\ &\leq C_1 \left[ \sum_1^{\infty} \left( \frac{A_n}{n} \right)^p n^{-2} K(n) \right]^{1/p} + C_1 \left[ \sum_1^{\infty} n^p a_n^p n^{-2} K(n) \right]^{1/p} + \\ &+ C_1 \left[ \sum_1^{\infty} n^p B_n^p n^{-2} K(n) \right]^{1/p} = S_1 + S_2 + S_3. \end{aligned}$$

To estimate  $S_1$  put  $c_n = na_n$  and  $K_1(n) = n^{-2}K(n)$ . By hypothesis we have  $\bar{\rho}(K) < 1 + p$ . Thus it follows that  $\bar{\rho}(K_1) = \bar{\rho}(K) - 2 < 1 + p - 2 = p - 1$ , so that  $K_1$  satisfies the assumptions of Lemma 1. Applying the second part of this lemma to  $S_1$  we obtain

$$(7) \quad \begin{aligned} S_1^p &= \sum_1^{\infty} n^{-2} K(n) \left( \frac{A_n}{n} \right)^p = \sum_1^{\infty} K_1(n) \left( \frac{A_n}{n} \right)^p \leq C \sum_1^{\infty} K_1(n) c_n^p = \\ &= C \sum_1^{\infty} n^{-2} K(n) n^p a_n^p. \end{aligned}$$

To estimate  $S_3$  put  $c_n = \frac{a_n}{n}$  and  $K_2(n) = n^{2p-2}K(n)$ . Since  $\underline{\rho}(K) > 1 - p$ , it follows that  $\underline{\rho}(K_2) = \underline{\rho}(K) + 2p - 2 > 1 - p + 2p - 2 = p - 1$ ; thus  $K_2$  satisfies the

assumptions of Lemma 2. Applying this lemma to  $S_3$  we obtain

$$(8) \quad \begin{aligned} S_3^p &= \sum_1^\infty n^{2p-2} K(n) \left( \frac{B_n}{n} \right)^p = \sum_1^\infty K_2(n) \left( \frac{B_n}{n} \right)^p \leq C \sum_1^\infty K_2(n) c_n^p = \\ &= C \sum_1^\infty n^{2p-2} K(n) \left( \frac{a_n}{n} \right)^p = C \sum_1^\infty n^{p-2} K(n) a_n^p. \end{aligned}$$

Finally, from (6), (7) and (8) it follows that

$$\left[ \int_0^\pi |g(x)|^p K\left(\frac{1}{x}\right) dx \right]^{1/p} \leq C \left[ \sum_1^\infty n^{p-2} K(n) a_n^p \right]^{1/p}.$$

This establishes part  $\Leftarrow$  of the theorem.

REMARK. If, like in [7], the sequence  $\{a_n\}$  is supposed to satisfy the additional assumption (\*), the proof of part  $\Leftarrow$  becomes somewhat shorter. Indeed, it is easily seen that in this case  $B_n \leq C a_n$ ; therefore  $S_3$  in (6) is estimated by  $S_2$ .

The proof of part  $\Rightarrow$  is very similar to the corresponding part in Theorem 1 [7] (assumption (\*) not being used at all for this part). The only difference is that we replace the class of regularly varying functions by the larger class of 0-regularly varying functions.

First note that  $K\left(\frac{1}{x}\right) g^p(x) \in L(0, \pi)$  implies  $g \in L(0, \pi)$ . Indeed,

$$(9) \quad \int_0^\pi |g(x)| dx \leq \left( \int_0^\pi x^{-\beta p} |g(x)|^p dx \right)^{1/p} \left( \int_0^\pi x^{\beta p'} dx \right)^{1/p'}$$

by Hölder's inequality. Chose  $\beta$  such that  $0 < \beta p < \rho$  (this is possible, since  $\rho > p - 1 > 0$ ). For such  $\beta$  both integrals in (9) converge (the first by property (iii-2) of 0-RV functions).

Now since the integrability of  $g$  implies that  $a_n$  are the Fourier coefficients of  $g$ , it follows that

$$G(x) = \int_0^x g(t) dt = \sum_{k=1}^\infty \frac{a_k}{k} (1 - \cos kx) = 2 \sum_{k=1}^\infty \frac{a_k}{k} \sin^2 \frac{kx}{2}.$$

Therefore

$$(10) \quad \begin{aligned} G\left(\frac{\pi}{n}\right) &= 2 \sum_{k=1}^\infty \frac{a_k}{k} \sin^2 \frac{k\pi}{2n} \geq 2 \sum_{k=\lceil \frac{n}{2} \rceil}^n \frac{a_k}{k} \sin^2 \frac{k\pi}{2n} \geq \\ &\geq 2 \sum_{k=\lceil \frac{n}{2} \rceil}^n \frac{a_k}{k} \left( \frac{2k\pi}{\pi 2n} \right)^2 \geq \frac{2}{4n} \sum_{k=\lceil \frac{n}{2} \rceil}^n a_k. \end{aligned}$$

Now according to the definitions of quasi-monotone sequences it follows that for  $\frac{n}{2} < k \leq n$

$$a_k \geq \frac{a_{k+1}}{1 + \frac{\alpha}{k}} \geq \dots \geq \frac{a_k}{\left(1 + \frac{\alpha}{k}\right)^{n-k}} \geq \frac{a_n}{\left(1 + \frac{\alpha}{k}\right)^k} \geq a_n e^{-\alpha}.$$

Hence (10) implies that

$$(11) \quad G\left(\frac{\pi}{n}\right) \geq \frac{1}{2n} \sum_{\left[\frac{n}{2}\right]+1}^n a_k \geq \frac{1}{2n} a_n e^{-\alpha} n = C a_n.$$

Denoting  $\psi(x) = \int_0^x |g(t)| dt$  we have

$$\begin{aligned} \sum_1^\infty n^{p-2} K(n) a_n^p &\leq C \sum_1^\infty n^{p-2} K(n) G^p\left(\frac{\pi}{n}\right) \leq && \text{[by (11)]} \\ &\leq C \sum_1^\infty n^{p-2} K(n) \psi^p\left(\frac{\pi}{n}\right) \leq C_1 \sum_1^\infty \psi^p\left(\frac{\pi}{n}\right) \int_{\pi/n}^{\pi/(n-1)} \left(\frac{1}{x}\right)^p K\left(\frac{1}{x}\right) dx && \text{[by (ii)]} \\ &\leq C_1 \sum_1^\infty \int_{\pi/n}^{\pi/(n-1)} \left(\frac{1}{x}\right)^p K\left(\frac{1}{x}\right) \psi^p(x) dx = C_1 \int_0^\pi K\left(\frac{1}{x}\right) \left(\frac{\psi(x)}{x}\right)^p dx \leq \\ &\leq C_2 \int_0^\pi K\left(\frac{1}{x}\right) |g(x)|^p dx \end{aligned}$$

by Corollary 2 (since  $\rho(K) > 1 - p$ ). This completes the proof of Theorem 1.

Note that in the proof of part  $\Rightarrow$  only condition  $\rho(K) > 1 - p$  is used; therefore no restriction on the upper index is necessary for this part of the theorem.

PROOF OF THEOREM 2. To prove part  $\Leftarrow$  consider

$$\begin{aligned} |f(x)| &= \left| \sum_{k=1}^\infty a_k \cos kx \right| \leq \sum_{k=1}^n a_k + \left| \sum_{k=n+1}^\infty a_k \cos kx \right| \leq \\ &\leq \sum_{k=1}^n a_k + \left| \frac{1}{2 \sin \frac{x}{2}} \sum_{k=n}^\infty (a_k - a_{k+1}) \left( \sin \left( n + \frac{1}{2} \right) x - \sin \left( k + \frac{1}{2} \right) x \right) \right| \leq \\ &\leq \sum_{k=1}^n a_k + \frac{\pi}{x} \sum_{k=n}^\infty |a_k - a_{k+1}|. \end{aligned}$$

If we set  $\frac{\pi}{n} < x \leq \frac{\pi}{n-1}$  and apply (3) we obtain

$$|f(x)| \leq \sum_{k=1}^n a_k + n \left( a_n + 2\alpha \sum_{k=n}^\infty \frac{a_k}{k} \right)$$

or, introducing the notation  $A_n = \sum_{k=1}^n a_k$ ,  $B_n = \sum_{k=n}^{\infty} \frac{a_k}{k}$ ,

$$|f(x)| \leq C(A_n + na_n + nB_n).$$

Now we have like in the proof of Theorem 1

$$(12) \quad \begin{aligned} & \left[ \int_0^\pi |f(x)|^p K\left(\frac{1}{x}\right) dx \right]^{1/p} + C \left[ \sum_1^\infty (A_n + na_n + nB_n)^p n^{-2} K(n) \right]^{1/p} \leq \\ & \leq C \left[ \sum_1^\infty A_n^p n^{-2} K(n) \right]^{1/p} + C \left[ \sum_1^\infty n^p a_n^p n^{-2} K(n) \right]^{1/p} + \\ & + C \left[ \sum_1^\infty n^p B_n^p n^{-2} K(n) \right]^{1/p} = \bar{S}_1 + S_2 + S_3. \end{aligned}$$

Since in estimating the sum  $S_3$  in (8) only condition  $\underline{\rho}(K) > 1 - p$  was used, it follows that (8) remains in this case also.

On the other hand, since  $\bar{\rho}(K) < 1$ , if we set  $K_3(n) = n^{p-2}K(n)$  it follows that  $\bar{\rho}(K_3) = \bar{\rho}(K) + p - 2 < 1 + p - 2 = p - 1$ . Therefore Lemma 1.2 may be applied to  $K_3$

$$(13) \quad \begin{aligned} \bar{S}_1^p &= \sum_1^\infty n^{p-2} K(n) \left(\frac{A_n}{n}\right)^p = \sum_1^\infty K_3(n) \left(\frac{A_n}{n}\right)^p \leq C \sum_1^\infty K_3(n) a_n^p = \\ &= C \sum_1^\infty n^{p-2} K(n) a_n^p. \end{aligned}$$

Finally, from (12), (8) and (13) it follows that

$$\left[ \int_0^\pi |f(x)|^p K\left(\frac{1}{x}\right) dx \right]^{1/p} \leq C \left[ \sum_1^\infty n^{p-2} K(n) a_n^p \right]^{1/p}$$

which establishes part  $\Leftarrow$  of the theorem.

To prove the converse, first note that  $f^p(x)K\left(\frac{1}{x}\right) \in L(0, \pi)$  implies  $f \in L(0, \pi)$  (cf. (9)). Therefore

$$F(x) = \int_0^x f(t) dt = \sum_1^\infty \frac{a_k}{k} \sin kx.$$

Denote  $\frac{a_k}{k}$  by  $b_k$ . It is easily seen that whenever  $\{a_n\}$  is quasi-monotone, then so is  $\{b_n\}$

$$b_{k+1} = \frac{a_{k+1}}{k+1} \leq \frac{a_k}{k+1} \left(1 + \frac{\alpha}{k}\right) \leq \frac{a_k}{k} \left(1 + \frac{\alpha}{k}\right) = b_k \left(1 + \frac{\alpha}{k}\right).$$



Thus we can apply part  $\Rightarrow$  of Theorem 1 to the function

$$F(x) = \sum_1^{\infty} b_k \sin kx$$

and conclude that for 0-RV functions  $K_4$  such that  $\underline{\rho}(K_4) > 1 - p$  the following holds

$$\sum_1^{\infty} n^{p-2} K_4(n) b_n^p \leq C \int_0^{\pi} K_4\left(\frac{1}{x}\right) |F(x)|^p dx.$$

Putting  $K_4(n) = n^p K(n)$  (for which  $\underline{\rho}(K_4) = p + \underline{\rho}(K) > \underline{\rho}(K) > 1 - p$ ) we obtain

$$(14) \quad \sum_1^{\infty} n^{p-2} n^p K(n) b_n^p \leq C \int_0^{\pi} K\left(\frac{1}{x}\right) \frac{|F(x)|^p}{x^p} dx.$$

Therefore, since  $|F(x)| \leq \int_0^x |f(t)| dt \equiv \psi(x)$ , relation (14) yields

$$\sum_1^{\infty} n^{p-2} K(n) a_n^p \leq C \int_0^{\pi} K\left(\frac{1}{x}\right) \frac{\psi^p(x)}{x^p} dx.$$

Finally we can apply Corollary 1 to the last integral to obtain

$$\sum_1^{\infty} n^{p-2} K(n) a_n^p \leq C_1 \int_0^{\pi} K\left(\frac{1}{x}\right) |f(x)|^p dx$$

which completes the proof of the theorem.

#### 4. Proofs of the lemmas

PROOF OF LEMMA 1. Since by hypothesis  $\bar{p} < p - 1$  and  $p \geq 1$ , it follows that  $\bar{p} < 0$ . Thus it is possible to choose  $\lambda$  such that  $\bar{p} < \lambda p < 0$ .

Now we have by Hölder's inequality

$$\begin{aligned} F(t) &= \int_a^t f(u) du \leq \left( \int_a^t u^{\lambda p} f^p(u) du \right)^{1/p} \left( \int_a^t u^{-\lambda p'} du \right)^{1/p'} \leq \\ &\leq (C t^{-\lambda p' + 1})^{1/p'} \left( \int_a^t u^{\lambda p} f^p(u) du \right)^{1/p} \end{aligned}$$

or

$$F^p(t) \leq C_1 t^{-\lambda p + \frac{p}{p'}} \int_a^t u^{\lambda p} f^p(u) du.$$

This implies that

$$\begin{aligned} \int_a^\infty K(t) \frac{F^p(t)}{t^p} dt &\leq C_1 \int_a^\infty K(t) t^{-p} t^{-\lambda p + \frac{p}{p'}} \int_a^t u^{\lambda p} f^p(u) du dt = \\ (15) \quad &= C_1 \int_a^\infty u^{\lambda p} f^p(u) \int_u^\infty K(t)^{-\lambda p - 1} dt du. \end{aligned}$$

Now since  $\lambda$  is chosen such that  $\lambda p > \bar{p}$ , we can apply property (i-2) of 0-RV functions to the inner integral in (15) and obtain

$$\int_a^\infty K(t) \frac{F^p(t)}{t^p} dt \leq C_2 \int_a^\infty u^{\lambda p} f^p(u) K(u) u^{-\lambda p} du = C_2 \int_a^\infty f^p(u) K(u) du.$$

This establishes part 1) of the lemma. It is obvious that the proof of part 2) follows along the same lines.

The proof of Lemma 2 is very similar to the proof of Lemma 1 (except that property (i-1) of 0-RV functions has to be used instead of property (i-2)), and we therefore omit it.

Corollaries 1 and 2 follow from Lemma 1 and 2 respectively, by introducing an obvious change of variables in the integrals.

#### REFERENCES

- [1] S. Aljančić, R. Bojanić, M. Tomić, *Sur l'intégrabilité de certaines séries trigonométriques*, Publ. Inst. Math. 8 (1955) 67–84.
- [2] S. Aljančić, D. Arandelović, *0-regularly varying functions*, Publ. Inst. Math. 22 (36) (1977) 5–22.
- [3] S. Aljančić, *Some applications of 0-regularly varying functions*, to appear.
- [4] R.P. Boas, *Integrability theorems for trigonometric transforms*, Springer-Verlag Berlin 1967.
- [5] S. Igari, *Some integrability theorems of trigonometric series and monotone decreasing functions*, Tôhoku Math. J. 12 (1960) 139–146.
- [6] S.M. Shah, *Trigonometric series with quasi-monotone coefficients*, Proc. Amer. Math. Soc. 13 (1962) 266–273.
- [7] C.-H. Yong, *Integrability theorems of power of trigonometric series with quasi-monotone coefficients*, Math. Zeitschr. 91 (1966) 280–293.
- [8] C.-H. Yong, *Asymptotic behaviour of trigonometric series with modified monotone coefficients*, The Chinese University of Hong Kong, 1974.